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Solving New Type of Linear Partial Differential Equations by Using New Transformation

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Abstract
Our aim in this paper is to introduce a new transform is known Al-Zughair transform for a function and how to use it to solve ODEs and PDEs.

Keywords: Linear Partial Differential Equations, a new transform, Al-Zughair transform

Introduction

Al-Zughair transform plays an important role to solve ODE and PDE with variable coefficients and this transformation appeared for the first time at 2017 [2].

Preliminaries

Definition (1) [1]:

Let f is defined function at period (a, b) then the integral transformation for f who's its symbol $F(s)$ is defined as:

$$F(s) = \int_a^b k(s,x)\, f(x)\, dx$$

Where k is a fixed function of two variables, called the kernel of the transformation and a, b are real numbers or $\mp\infty$, such that the above integral is convergent.

Definition (2) [2]: Al-Zughair transformation $\left[Z\big(f(x)\big)\right]$ for the function $f(x)$ where $x \in [1, e]$ is defined by the following integral:

$$Z\big(f(x)\big) = \int_1^e \frac{(\ln x)^s}{x} f(x)dx = F(s)$$

Such that this integral is convergent, s is positive constant. From the above definition we can write:

$$Z\big(u(x,t)\big) = \int_1^e \frac{(\ln t)^s}{t} u(x,t)dt = v(x,s)$$

Such that $u(x,t)$ is a function of x and t.

Property (1): (Linear property)

This transformation is characterized by the linear property, that is

$$Z[Au_1(x,t) \pm Bu_2(x,t)] = AZ[u_1(x,t)] \pm BZ[u_2(x,t)]$$

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Where A and B are constants while, the functions $u_1(x,t)$, $u_2(x,t)$ are defined when $t \in [1,e]$.

Proof:

$$Z[Au_1(x,t) \pm Bu_2(x,t)] = \int_1^e \frac{(ln\,t)^s}{t}\big(Au_1(x,t) \pm Bu_2(x,t)\big)dt$$

$$= \int_1^e \frac{(ln\,t)^s}{t} Au_1(x,t)dt \pm \int_1^e \frac{(ln\,t)^s}{t} Bu_2(x,t)dt$$

$$= A\int_1^e \frac{(ln\,t)^s}{t} u_1(x,t)dt \pm B\int_1^e \frac{(ln\,t)^s}{t} u_2(x,t)dt$$

$$= AZ[u_1(x,t)] \pm BZ[u_2(x,t)]$$

Al-Zughair transformation for some fundamental functions is given in table (1) [2]:

ID	Function, $f(x)$	$F(s) = \int_1^e \frac{(ln\,x)^s}{x} f(x)dx = Z(f(x))$	Regional of convergence
1	k ; k = constant	$\frac{k}{(s+1)}$	$s > -1$
2	$(\ln x)^n, n \in R$	$\frac{1}{(s+(n+1))}$	$s > -(n+1)$
3	$\ln(\ln x)$	$\frac{-1}{(s+1)^2}$	$s > -1$
4	$(\ln(\ln x))^n, n \in z^+$	$\frac{(-1)^n n!}{(s+1)^{n+1}}$	$s > -1$
5	$sin(a\,ln(ln\,x))$	$\frac{-a}{(s+1)^2+a^2}$	$s > -1$ a is constant
6	$cos(a\,ln(ln\,x))$	$\frac{s+1}{(s+1)^2+a^2}$	$s > -1$ a is constant
7	$sinh(a\,ln(ln\,x))$	$\frac{-a}{(s+1)^2-a^2}$	$\lvert s+1 \rvert > a$ a is constant
8	$cosh(a\,ln(ln\,x))$	$\frac{s+1}{(s+1)^2-a^2}$	$\lvert s+1 \rvert > a$ a is constant

From Al-Zughair transform definition and the above table, we get:

Theorem (1):
If $Z(u(x,t)) = v(x,s)$ and a is constant, then $Z((ln\,t)^a u(x,t)) = v(x,s+a)$.

Proof:

$$Z((ln\,t)^a u(x,t)) = \int_1^e \frac{(ln\,t)^s}{t}(ln\,t)^a u(x,t)dt$$

$$= \int_1^e \frac{(ln\,t)^{s+a}}{t} u(x,t)dt = v(x,s+a) \blacksquare$$

Example (1): By using the table (1) of Al-Zughair transformation we will consider that:
$f(x,t) = 4x^2 \ln t + x^3$

$$Z(f(x,t)) = \int_1^e \frac{(\ln t)^s}{t} f(x,t)dt$$

$$= \int_1^e \frac{(ln\,t)^s}{t}(4x^2\,ln\,t + x^3)dt$$

$$= 4x^2\int_1^e \frac{(ln\,t)^{s+1}}{t}dt + x^3\int_1^e \frac{(ln\,t)^s}{t}dt$$

$$= 4x^2 \frac{(\ln t)^{s+2}}{s+2}\Bigg|_1^e + x^3\frac{(ln\,t)^{s+1}}{s+1}\Bigg|_1^e = \frac{4x^2}{s+2} + \frac{x^3}{s+1}$$

Example (2): To find Al-zughair transform of
$f(x,t) = \ln x\,(\ln t)^3 + x \sin \ln(\ln t)$
$Z(f(x,t)) = Z(ln\,x\,(ln\,t)^3 + x\,sin\,ln(ln\,t))$
$= Z(ln\,x\,(ln\,t)^3) + Z(x\,sin\,ln(ln\,t))$
$= \frac{\ln x}{s+4} - \frac{x}{(s+1)^2+1}$

Definition (3) [2]:
Let $u(x,t)$ be a function where $t \in [1,e]$ and $Z(u(x,t)) = v(x,s)$, $u(x,t)$ is said to be an inverse for the Al-Zughair transformation and written as $\mathcal{T}^{-1}(v(x,s)) = u(x,t)$, where Z^{-1} returns the transformation to the original function. For example

1) $Z^{-1}\left[\frac{-sinx}{(s+1)^2}\right] = \ln(\ln t)\, sinx \quad , \; s > -1.$
2) $Z^{-1}\left[\frac{x}{s+5}\right] = x(\ln t)^4 \quad , \; s > -5.$
3) $Z^{-1}\left[\frac{sinx\,(s+1)}{(s+1)^2-4}\right] = \sin x \;\; \cosh(2\ln(\ln t)) \quad ,$ $|s+1| > 2$.

Property (2): If $Z^{-1}(v_1(x,s)) = u_1(x,t)$, $Z^{-1}(v_2(x,s)) = u_2(x,s)$, $\ldots, Z^{-1}(v_n(x,t)) = u_n(x,s)$ and $a_1, a_1, \ldots a_n$ are constants then,

$$Z^{-1}[a_1v_1(x,s) + a_2v_2(x,s) + \cdots + a_nv_n(x,s)] = a_1u_1(x,t) + a_2u_2(x,t) + \cdots + a_n\, u_n(x,t)$$

Solving New Type of Linear Partial Differential Equations by Using New Transformation
Definition (4):
The equation

$$a_o(\ln t)^n u_t^{(n)}(x,t) + a_1(\ln t)^{n-1}u_t^{(n-1)}(x,t) + \cdots + a_{n-1}(\ln t)u_t(x,t) + a_nu(x,t) = f(x,t)$$

Where a_o, a_1, …, a_n are constants and $f(x,t)$ is a function of x and t, we will call it ***Ali's Equation*** in partial Differential equation .

Theorem (2):
If the function $u(x,\ln t)$ is defined for $t \in [1,e]$ and its derivatives $u_t(x,\ln t), u_{tt}(x,\ln t), \ldots, u_t^{(n)}(x,\ln t)$ are exist then:

$$Z[(\ln t)^n u_t^{(n)}(x,\ln t)] = u_t^{(n-1)}(x,1) + (-1)^n(s+n)u_t^{(n-2)}(x,1) + (-1)^{n-1}(s+n)(s+(n-1))u_t^{(n-3)}(x,1) + \cdots + (s+n)(s+\;(n-1))\ldots(s+2)u_t(x,1) \;+ (-1)^n(s+n)(s+\;(n-1))\ldots(s+2)(s+1)v(x,s).$$

Proof:
If $n = 1$

$$Z(\ln t\, u_t(x,\ln t)) = \int_1^e \frac{(\ln t)^s}{t}(\ln t)u_t(x,\ln t)dt = \int_1^e \frac{(ln\, t)^{s+1}}{t}u_t(x,\ln t)dt$$

Let $y = (\ln t)^{s+1} \Rightarrow dy = (s+1)\frac{(ln\,t)^s}{t}dt$

$$dh = \frac{u_t(x,\ln t)}{t}dt \;\Rightarrow h = u(x,\ln t)$$

$$\int_1^e \frac{(ln\,t)^{s+1}}{t}u_t(x,\ln t)dt = (ln\,t)^{s+1}u(x,ln\,t)|_1^e - (s+1)\int_1^e \frac{(ln\,t)^s}{t}u(x,lnt)dt = u(x,1) - (s+1)Z(u(x,\ln t))$$

If $n = 2$

$$Z((\ln t)^2u_{tt}(x,\ln t)) = \int_1^e \frac{(ln\,t)^{s+2}}{t}u_{tt}(x,\ln t)dt$$

Let $y = (\ln t)^{s+2} \Rightarrow dy = (s+2)\frac{(ln\,t)^{s+1}}{t}dt$

$$dh = \frac{u_{tt}(x,\ln t)}{t}dt \;\Rightarrow h = u_t(x,\ln t)$$

$$\begin{aligned}\int_1^e \frac{(ln\,t)^{s+2}}{t}u_{tt}(x,\ln t)dt &= (ln\,t)^{s+2}u_t(x,ln\,t)|_1^e - (s+2)\int_1^e \frac{(ln\,t)^{s+1}}{t}u_t(x,lnt)dt \\ &= u_t(x,1) - (s+2)Z(\ln t\, u_t(x,\ln t)) \\ &= u_t(x,1) - (s+2)Z(\ln t\, u_t(x,ln\,t)) \\ &= u_t(x,1) - (s+2)u(x,1) + (s+2)(s+1)Z(u(x,ln\,t))\end{aligned}$$

If $n = 3$

$$Z((\ln t)^3u_{ttt}(x,\ln t)) = \int_1^e \frac{(ln\,t)^{s+3}}{t}u_{ttt}(x,\ln t)dt$$

$$y = (\ln t)^{s+3} \;\;\Rightarrow\;\; dy = (s+3)\frac{(ln\,t)^{s+2}}{t}dt$$

$$dh = \frac{u_{ttt}(x,ln\,t)}{t}dt \;\;\Rightarrow\;\; h = u_{tt}(x,ln\,t)$$

$$\begin{aligned}\int_1^e \frac{(ln\,t)^{s+3}}{t}u_{ttt}(x,ln\,t)dt &= (ln\,t)^{s+3}u_{tt}(x,ln\,t)|_1^e - (s+3)\int_1^e \frac{(ln\,t)^{s+2}}{t}u_{tt}(x,lnt)dt \\ &= u_{tt}(x,1) - (s+3)Z((ln\,t)^2u_{tt}(x,ln\,t)) \\ &= u_{tt}(x,1) - (s+3)u_t(x,1) + (s+3)(s+2)u(x,1) - (s+3)(s+2)(s+1)Z(u(x,\ln t)).\end{aligned}$$

And so on,

$$Z[(\ln t)^n u_t^{(n)}(x,\ln t)] = u_t^{(n-1)}(x,1) + (-1)^n(s+n)u_t^{(n-2)}(x,1) + (-1)^{n-1}(s+n)(s+(n-1))u_t^{(n-3)}(x,1) + \cdots + (s+n)(s+\;(n-1))\ldots(s+2)u_t(x,1) + (-1)^n(s+n)(s+\;(n-1))\ldots(s+2)(s+1)v(x,s) \blacksquare.$$

Example (1): To solve the differential equation
$\ln t\, u_t(x,\ln t) - 3u(x,\ln t) = xsin(2\ln(\ln t))$; $u(x,1) = -5$
we take Z-transform to both sides of above equation we get :

$$Z[\ln t\, u_t(x,\ln t)] - 3Z[u(x,\ln t)] = xZ[sin(2\ln(\ln t))]$$

$$u(x,1) - (s+1)Z[u(x,ln\,t)] - 3Z[u(x,ln\,t)] = \frac{-2x}{(s+1)^2+4}$$

$$-5 - (s+4)Z[u(x,ln\,t)] = \frac{-2x}{(s+1)^2+4}$$

$$Z[u(x,ln\,t)] = \frac{2x}{(s+4)((s+1)^2+4)} - \frac{5}{(s+4)}$$

By take Z^{-1}-transform to both side of above equation we get:

$$u(x,t) = Z^{-1}\left[\frac{A(x)(s+1)+B(x)}{((s+1)^2+4)} + \frac{C(x)}{(s+4)}\right] - 5(ln\,t)^3$$

$$A(x) = \frac{-2x}{13} \quad , \quad B(x) = \frac{6x}{13} \quad , \quad C(x) = \frac{2x}{13}$$

$$u(x,\ln t) = Z^{-1}\left[\frac{-2x}{13}\frac{(s+1)}{((s+1)^2+4)}\right] + Z^{-1}\left[\frac{6x}{13}\frac{1}{((s+1)^2+4)}\right] + Z^{-1}\left[\frac{2x}{13}\frac{1}{(s+4)}\right] - 5(ln\,t)^3$$

$$= \frac{-2x}{13}\cos(2\,ln(ln\,t)) - \frac{6x}{26}\sin(2\ln(\ln t)) + \frac{2x}{13}(\ln t)^3 - 5(ln\,t)^3$$

Example (2): To find the solution of the differential equation

$$(\ln t)^2 u_{tt}(x,\ln t) - \ln t\, u_t(x,\ln t) + u(x,\ln t) = \ln(\ln t)\sin x \;;$$

$$u(x,1) = 1 \;, \qquad u_t(x,1) = 3$$

we take Z-transform to both sides of above equation we get :

$$Z[(\ln t)^2 u_{tt}(x,\ln t)] - Z[\ln t\, u_t(x,\ln t)] + Z[u(x,\ln t)] = sin\,x\,Z[ln(ln\,t)]$$

$$u_t(x,1) - (s+2)u(x,1) + (s+2)(s+1)Z[u(x,ln\,t)] - u(x,1) + (s+1)Z[u(x,ln\,t)] + Z[u(x,ln\,t)] = \frac{-\sin x}{(s+1)^2}$$

$$-s + (s+2)^2 Z[u(x,ln\,t)] = \frac{-\sin x}{(s+1)^2}$$

$$Z[u(x,ln\,t)] = \frac{-\sin x}{(s+1)^2(s+2)^2} + \frac{s}{(s+2)^2}$$

$$= \frac{-\,sin\,x}{(s+1)^2(s+2)^2} + \frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}$$

By take Z^{-1}-transform to both side of above equation we get:

$$u(x,\ln t) = Z^{-1}\left[\frac{-\,sin\,x}{(s+1)^2(s+2)^2}\right] + \ln t + 2(\ln t)(\ln(ln\,t))$$

$$= \frac{A(x)(s+1)+B(x)}{(s+1)^2} + \frac{C(x)(s+2)+D(x)}{(s+2)^2} - ln\,t + 2\,ln(ln\,t)$$

$$A(x) = 2\sin x\,, B(x) = -\sin x\;, C(x) = -2\sin x\;, D(x) = -\sin x$$

$$u(x,\ln t) = 2\sin x + (\ln(\ln t))\sin x - 2(\ln t)\sin x + (\ln t)(\ln(\ln t))\sin x + \ln t + 2(\ln t)(\ln(\ln t))$$

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