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Solving Of Third Order Retarded Dynamical System via Lambert W Function and Stability Analysis

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Abstract
In this paper, we use the Lambert W function to derive a solution third order dynamical systems of retarded type. As well as we discuss analytical stability through illustrative examples.

Keywords: Delay differential equations, Lambert W function, stability

1. Introduction

The Lambert W function grows in the modeling of phenomena in numerous fields of science and engineering. In physics, it is used for the purpose of quantum theory, plasma physics and solar physics and many other applications. (See, [6], [3] [7] and [4].)
Many researchers' works on the Lambert W function to obtain analytical solutions of delay differential equations (see, [8], [3], and [2]). Also, many researchers are concerned with studying the stability of delay differential equations besides studying their solutions their (see, [1], [8], [9], [5]) the infinite branches the Lambert W function are defined in the following series [3]

$$W_k(z) = \ln_k(z) - \ln(\ln_k(z)) + \sum_{i=0}^{\infty}\sum_{m=1}^{\infty} C_{im} \frac{(\ln(\ln_k(z)))^m}{(\ln_k(z))^{m+1}} \qquad (1)$$

Where $\ln_k(z)$ are the k^{th} logarithm branch and the coefficients C_{im} can be expressed in terms of nonnegative sterling numbers of first kind.
The principal branch (i.e., k = 0) of the Lambert W function can be represented by the following power series

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n$$

Where $z \in C$ [3].

Theorem (1.1) [3]. For any given $z \in R$, the principal branch $W_0(z)$ of the Lambert W function defined by

$$W_0(z) = \begin{cases} x, & \forall x \geq -1, \quad where\ z \geq \frac{-1}{e} \\ -y\cot gy + iy, & \forall y \in (0,\pi) \quad where\ z < \frac{-1}{e} \end{cases}$$

Have the following properties:

1. $W_0(z)$ Is real and increasing if $z \in (\frac{-1}{e}, \infty)$.

2. $W_0(z)$ Is complex valued and decreasing negative real part if $z \in (\frac{-\pi}{2}, \frac{-1}{e})$.
3. $W_0(z)$ Is complex and purely imaginary parts if $z = \frac{-\pi}{2}$.
4. $W_0(z)$ Is complex valued with decreasing positive real part if $z \in (-\infty, \frac{-\pi}{2})$.

2. Solution of Retarded Dynamical Systems of Third Order and their stability

Consider the following third order retarded dynamical system with constant coefficients

$$\frac{d^3}{dt^3}y(t) + a_2\frac{d^2}{dt^2}y(t) + a_1\frac{d}{dt}y(t) + a_0 y(t) = by(t-T) \quad (2)$$

With the initial delay condition

$$y(t) = \varphi(t), \ -T \le t \le 0.$$

If we assume that $a_2 = 3\alpha, a_1 = 3\alpha^2$ and $a_0 = \alpha^3$, then equation (2) become (3)

We are concerned in finding the solution beside stability analysis for the equation (3) that's a time delay T.

Let λ be a complex number and the solution of equation (3) be of the formula $y(t) = e^{\lambda t}$. Substituting $y(t) = e^{\lambda t}$ in equation (3) gives (4)

Which represents the transcendental characteristic equation of delay differential equation (3)? Multiplying both sides of equation (4) by $e^{\lambda T}$ yields

$$(\lambda + \alpha)^3 e^{\lambda T} = b. \quad (5)$$

Taking the cubic root on both sides of equation (5) gives

$$(\lambda + \alpha)\, e^{\frac{\lambda T}{3}} = \sqrt[3]{b}. \quad (6)$$

Since, every function $W(\lambda)$ which satisfies $W(\lambda)e^{W(\lambda)} = \lambda$ can be expressed in terms of the Lambert W function [2]. Then the roots of the characteristic equation (4) can be found by writing equation (6) as follows:

$$(\frac{T\lambda}{3} + \frac{T\alpha}{3})e^{(\frac{T\lambda}{3}+\frac{T\alpha}{3})} = \frac{T}{3}e^{\frac{T\alpha}{3}}\sqrt[3]{b}. \quad (7)$$

Then

$$W(\frac{T}{3}e^{\frac{T\alpha}{3}}\sqrt[3]{b}) = \frac{T\lambda}{3} + \frac{T\alpha}{3} \quad (8)$$

Thus, the characteristic roots of equation (4) is

$$\lambda = \frac{3}{T}W(\frac{T}{3}e^{\frac{T\alpha}{3}}\sqrt[3]{b}) - \alpha \quad (9)$$

It is clear that the characteristic root λ depends on of the parameters α, T, b (in terms of a_2, a_1 and a_0). Hence, the stability property of DDE (3) depends on those parameters. We have the general solution of the delay differential equation (3) as follows

$$y(t) = \sum_{i=-\infty}^{\infty} K_i\, e^{\lambda_i t}$$

$$= \sum_{i=-\infty}^{\infty} K_i\, e^{[\frac{3}{T}W_i(\frac{T}{3}e^{\frac{T\alpha}{3}}\sqrt[3]{b})-\alpha]t}, \quad t \in [-T, 0] \quad (10)$$

Since, the solution $y(t)$ is equal to the initial delay condition $\phi(t)$ for $t \in [-T, 0]$, and then we have

$$\phi(t) = \sum_{i=-\infty}^{\infty} K_i\, e^{[\frac{3}{T}W_i(\frac{T}{3}e^{\frac{T\alpha}{3}}\sqrt[3]{b})-\alpha]t}, \quad t \in [-T, 0] \quad (11)$$

Now, we find the coefficients K_i by using the function $\phi(t)$ as follows:

The interval $[-T, 0]$ can be partitioned into *2M* parts such that *M* is a sufficiently large positive integer number.

$$[-T,0] = [-T, -T+\frac{T}{2M}] \cup [-T+\frac{T}{2M}, -T+\frac{2T}{2M}] \cup [-T+\frac{2T}{2M}, -T+\frac{3T}{2M}] \cup \dots \cup [\frac{-T}{2M}, 0]$$

Then, we find the values of the initial delay function $\varphi(t)$ at the endpoints of the above subintervals using (11) as follows

$$\varphi(0) = \cdots + K_{-M}y_{-M}(0) + \cdots + K_{-1}y_{-1}(0) + K_0 y_0(0) + K_1 y_1(0) + \cdots + K_M y_M(0)$$

$$\varphi(\frac{-T}{2M}) = \cdots + K_{-M}y_{-M}(\frac{-T}{2M}) + \cdots + K_{-1}y_{-1}(\frac{-T}{2M}) + K_0 y_0(\frac{-T}{2M}) + K_1 y_1(\frac{-T}{2M}) + \cdots + K_M y_M(\frac{-T}{2M})$$

$$\varphi(-T) = \cdots + K_{-M}y_{-M}(-T) + \cdots + K_{-1}y_{-1}(-T) + K_0 y_0(-T) + K_1 y_1(-T) + \cdots + K_M y_M(-T)$$

We can write the above equations in matrix form as:

$$\begin{bmatrix}\varphi(0)\\ \varphi(\frac{-T}{2M})\\ \vdots\\ \varphi(-T)\end{bmatrix}=\begin{bmatrix}y_{-M}(0) & \cdots & y_M(0)\\ y_{-M}(\frac{-T}{2M}) & \cdots & y_M(\frac{-T}{2M})\\ \vdots & \ddots & \vdots\\ y_{-M}(-T) & \cdots & y_M(-T)\end{bmatrix}\begin{bmatrix}K_{-M}\\ K_{-M+1}\\ \vdots\\ K_M\end{bmatrix}$$

$\varphi(T,M)\approx Y(T,M)\,K(M).$

Thus

$$K(M)\approx Y^{-1}(T,M)\varphi(T,M)).$$

The coefficients K_i are given by

$$K_i=\lim_{M\to\infty}(Y^{-1}(T,M)\varphi(T,M))_i. \qquad (12)$$

To find the initial delay function $\varphi(t)$, we assume that $y(t)=0,\ t<-T$, then $y(t)=0,\ \forall t\in[-T,0]$. Then the equation (3) gives the following third order ODE

$$\frac{d^3}{dt^3}y(t)+3\alpha\frac{d^2}{dt^2}y(t)+3\alpha^2\frac{d}{dt}y(t)+\alpha^3 y(t)=0\,,\quad t>0 \qquad (13)$$

We shall use the general solution of equation (13) to be the required initial delay function $\varphi(t)$ in $[-T,0]$.

3. Illustrative Examples

Example (3.1): Consider the following third order DDE

$$\frac{d^3}{dt^3}y(t)+6\frac{d^2}{dt^2}y(t)+12\frac{d}{dt}y(t)+8y(t)=\ y(t-1),\ t>0 \qquad (14)$$

$$y(t)=\phi(t),\quad t\in[-1,0]$$

By comparison with equation (3), we have $b=1,\alpha=2$ and $T=1$. To consider the stability margin, we consider $y(t)=0,\ t<-1$ then $y(t\text{-}1)=0,\ \forall t\in[-1,0]$.

Hence the equation (14) becomes the third order ODE.

$$\frac{d^3}{dt^3}y(t)+6\frac{d^2}{dt^2}y(t)+12\frac{d}{dt}y(t)+8y(t)=\ 0 \qquad (15)$$

The general solution of equation (15) is given as

$$y(t)=(At^2+Bt+C)e^{-2t} \qquad (16)$$

Assuming $y(0)=\ 0$, $y(1)=\ 1$ and $y(2)=\ 1$ yields $A=19.91,\ B=\text{-}12.521$ and $C=0$. Hence the initial delay function of the equation (14) is

$$\varphi\ (t)=(19.91\,t^2-12.521t)e^{-2t},\ t\in[-1,0] \quad (17)$$

The general solution of third order DDE (14) is as follows

$$y(t)=\sum_{i=-\infty}^{\infty}K_i e^{[3W_i(\frac{1}{3}\ e^{\frac{2}{3}}\sqrt[3]{1})-2]t},\quad t\in[-1,0] \quad (18)$$

Where K_i unknown constants and the characteristic roots are are given by equation (9) as

$$\lambda=3W_i\ (\frac{1}{3}\ e^{\frac{2}{3}}\sqrt[3]{1})-2. \qquad (19)$$

Case (1)

$$y(t)=\sum_{i=-\infty}^{\infty}K_i\ e^{[3W_i\ (\frac{1}{3}e^{\frac{2}{3}})\ -2]t},\quad t\in[-1,0],$$

$$\lambda_i\ =3W_i\ (\frac{1}{3}\ e^{\frac{2}{3}})-2$$

To find the coefficients K_i we partition the interval $[-1,0]$ into 2 subintervals as follows.

$[\text{-}1,0]=[-1,\text{-}\ 0.5]\cup[\text{-}0.5,0]$

Then

$$\varphi(0)=K_{-1}y_{-1}(0)+K_0y_0(0)+K_1y_1(0)$$
$$\varphi(-0.5)=K_{-1}y_{-1}(-0.5)+K_0y_0(-0.5)+K_1y_1(-0.5)$$
$$\varphi(-1)=K_{-1}y_{-1}(-1)+K_0y_0(-1)+K_1y_1(-1)$$

The above system can be written matrix form as follows

$$\begin{bmatrix}\varphi(0)\\ \varphi(-0.5)\\ \varphi(-1)\end{bmatrix}=\begin{bmatrix}y_{-1}(0) & y_0(0) & y_1(0)\\ y_{-1}(-0.5) & y_0(-0.5) & y_1(-0.5)\\ y_{-1}(-1) & y_0(-1) & y_1(-1)\end{bmatrix}\begin{bmatrix}K_{-1}\\ K_0\\ K_1\end{bmatrix}$$

$$\begin{bmatrix}K_{-1}\\ K_0\\ K_1\end{bmatrix}=\begin{bmatrix}y_{-1}(0) & y_0(0) & y_1(0)\\ y_{-1}(-0.5) & y_0(-0.5) & y_1(-0.5)\\ y_{-1}(-1) & y_0(-1) & y_1(-1)\end{bmatrix}^{-1}\begin{bmatrix}\varphi(0)\\ \varphi(-0.5)\\ \varphi(-1)\end{bmatrix}$$

$$=\begin{bmatrix}1 & 1 & 1\\ 52.7282+7.1313i & 1.4377 & 52.7282\text{-}7.1313i\\ 2.7294e+03+7.5204e+02i & 2.0671 & 2.7294e+03\text{ -}7.5204e+02i\end{bmatrix}^{-1}$$

$$.\begin{bmatrix}0\\ 30.5481\\ 239.6345\end{bmatrix}$$

$$= \begin{bmatrix} 0.5560 + 1.8570i \\ -1.1120 - 0.0000i \\ 0.5560 - 1.8570i \end{bmatrix}$$

Therefore the solution of the equation (14) is given as

$$y(t) = (0.5560 + 1.8570i)\, e^{(-7.9484-12.8352i)t} + (-1.1120 - 0.0000i)\, e^{(-0.7261)t} + (0.5560 - 1.8570i) e^{(-7.9484+12.8352i)t}$$

The characteristic roots λ for the branches $i = 0, \pm 1, \pm 2, \pm 3,$ are as follows:

$i = 0; \quad \lambda = -0.7261$
$i = 1; \quad \lambda = -7.9484 + 12.8352i$
$i = -1; \quad \lambda = -7.9484 - 12.8352i$
$i = 2; \quad \lambda = -10.5183 + 32.2111i$
$i = -2; \quad \lambda = -10.5183 - 32.2111i$
$i = 3; \quad \lambda = -11.8656 + 51.2659i$
$i = -3; \quad \lambda = -11.8656 - 51.2659i$

From the above characteristic roots we note that $\mathrm{Re}(\lambda) < 0, \ \forall \lambda$, then equation (14) is stable.

Case (2)

$$y(t) = \sum_{i=-\infty}^{\infty} K_i e^{[3W_i(\frac{1}{3}e^{\frac{2}{3}}(\frac{-1}{2} + \frac{\sqrt{3}}{2}i)) - 2]t}, \quad t \in [-1, 0],$$

$$\lambda_i = 3W_i\left(\frac{1}{3}e^{\frac{2}{3}}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)\right) - 2.$$

$$\begin{bmatrix} K_{-1} \\ K_0 \\ K_1 \end{bmatrix} = \begin{bmatrix} y_{-1}(0) & y_0(0) & y_1(0) \\ y_{-1}(-0.5) & y_0(-0.5) & y_1(-0.5) \\ y_{-1}(-1) & y_0(-1) & y_1(-1) \end{bmatrix}^{-1} \begin{bmatrix} \varphi(0) \\ \varphi(-0.5) \\ \varphi(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ -19.6762 + 1.4878i & 1.5203 - 1.9907i & -90.5067 + 24.0569i \\ 3.8494e{+}002 - 5.8549e{+}001i & -1.6514 - 6.0531i & 7.6127e{+}003 - 4.3546e{+}003i \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 30.5481 \\ 239.6345 \end{bmatrix}$$

$$\begin{bmatrix} K_{-1} \\ K_0 \\ K_1 \end{bmatrix} = \begin{bmatrix} -1.8423 - 0.4647i \\ 0.2245 + 0.0485i \\ 0.0092 + 0.0064i \end{bmatrix}$$

Therefore the solution of the equation (14) is given as

$$y(t) = (-1.8423 - 0.4647i)e^{(-5.9645-6.1322i)t} + (0.2245 + 0.0485i)e^{(-1.8365+1.8371i)t} + (0.0092 + 0.0064i)e^{(-9.0791+19.3691i)t}$$

The characteristic roots λ for the branches $i = 0, \pm 1, \pm 2, \pm 3,$ are as follows:

$i = 0; \quad \lambda = -1.8365 + 1.8371i$
$i = 1; \quad \lambda = -9.0791 + 19.3691i$
$i = -1; \quad \lambda = -5.9645 - 6.1322i$
$i = 2; \quad \lambda = -11.0383 + 38.5795i$
$i = -2; \quad \lambda = -9.8866 - 25.8140i$
$i = 3; \quad \lambda = -12.2066 + 57.5933i$
$i = -3; \quad \lambda = -11.4806 - 44.9292i$

Since $\mathrm{Re}(\lambda) < 0, \ \forall \lambda$, then equation (14) is stable.

Case (3)

$$y(t) = \sum_{i=-\infty}^{\infty} K_i e^{[3W_i(\frac{1}{3}e^{\frac{2}{3}}(\frac{-1}{2} - \frac{\sqrt{3}}{2}i)) - 2]t}, \quad t \in [-1, 0],$$

$$\lambda_i = 3W_i\left(\frac{1}{3}e^{\frac{2}{3}}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)\right) - 2$$

$$\begin{bmatrix} K_{-1} \\ K_0 \\ K_1 \end{bmatrix} = \begin{bmatrix} y_{-1}(0) & y_0(0) & y_1(0) \\ y_{-1}(-0.5) & y_0(-0.5) & y_1(-0.5) \\ y_{-1}(-1) & y_0(-1) & y_1(-1) \end{bmatrix}^{-1} \begin{bmatrix} \varphi(0) \\ \varphi(-0.5) \\ \varphi(-1) \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 & 1 \\ -90.5067-24.0569i & 1.5203+1.9907i & -19.6762-1.4878i \\ 7.6127e+003+4.3546e+003i & -1.6514+6.0531i & 3.8494e+002+5.8549e+001i \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 30.5481 \\ 239.6345 \end{bmatrix}$$

$$=\begin{bmatrix} 0.0092-0.0064i \\ 1.7503-0.4012i \\ -1.8423+0.4647i \end{bmatrix}$$

Therefore the solution of the equation (14) is given as

$$y(t)=(0.0092-0.0064i)e^{(-9.0791-19.3691i)t}+(1.7503-0.4012i)e^{(-1.8365-1.8371i)t}+(-1.8423+0.4647i)e^{(-5.9645+6.1322i)t}$$

The characteristic roots λ_i for the branches $i=0,\ \pm1,\ \pm2,\ \pm3,$ are as follows:

$i=0;\quad \lambda=-1.8365-1.8371i$
$i=1;\quad \lambda=-5.9645+6.1322i$
$i=-1;\quad \lambda=-9.0791-19.3691i$
$i=2;\quad \lambda=-9.8866+25.8140i$
$i=-2;\quad \lambda=-11.0383-38.5795i$
$i=3;\quad \lambda=-11.4806+44.9292i$
$i=-3;\quad \lambda=-12.2066-57.5933i$

From the above characteristic roots we note that $\mathrm{Re}(\lambda)<0,\ \forall\lambda$, then equation (14) is stable.

Example (4.2): Consider the following third order DDE

$$\frac{d^3}{dt^3}y(t)-9\frac{d^2}{dt^2}y(t)+27\frac{d}{dt}y(t)-27y(t)=-2\,y(t\text{-}2),\quad t>0 \qquad (18)$$

$$y(t)=\phi(t),\qquad t\in[-2,0]$$

We have $b=-2, \alpha=-3$ and $T=2$. To consider the stability margin, we consider $y(t)=0, t<-2$ then $y(t\text{-}2)=0,\ \forall t\in[-2,0]$.

Therefore the equation (18) becomes the third order ODE.

$$\frac{d^3}{dt^3}y(t)-9\frac{d^2}{dt^2}y(t)+27\frac{d}{dt}y(t)-27y(t)=0 \qquad (19)$$

The general solution of equation (19) is given as

$$y(t)=(At^2+Bt+C)\,e^{3t} \qquad (20)$$

Assuming that $y(0)=0, y(0.5)=1$, and $y(1)=2$ we get $A=-0.6934,\ B=0.7929$ and $C=0$. Hence the initial delay function of the equation (18) is

$$\varphi(t)=(-0.6934t^2+0.7929t)\,e^{3t} \qquad (21)$$

The general solution of the third order DDE (18) by as follows by equation (10), we get

$$y(t)=\sum_{i=-\infty}^{\infty}K_i\,e^{[\frac{3}{2}W_i(\frac{2}{3}e^{-2}\sqrt[3]{-2})+3]t},\qquad t\in[-2,0]$$

Where K_i unknown constants and the characteristic roots are are given by equation (9) as

$$\lambda_i=W_i(\frac{2}{3}e^{-2}\sqrt[3]{-2})+3 \qquad (19)$$

Case (1)

$$y(t)=\sum_{i=-\infty}^{\infty}K_i\,e^{[\frac{3}{2}W_i(\frac{2}{3}e^{-2}\sqrt[3]{2}(\frac{1}{2}+\frac{\sqrt{3}}{2}i))+3]t},\qquad t\in[-2,0]$$

$$,\lambda_i=\frac{3}{2}W_i(\frac{2}{3}e^{-2}\sqrt[3]{2}(\frac{1}{2}+\frac{\sqrt{3}}{2}i))+3$$

To find the coefficients K_i we partition the interval $[-2,0]$ in to subintervals as follows.

$$[-2,0]=[-2,-1.5]\cup[-1.5,-1]\cup[-1,-0,5]\cup[-0,5,0]$$

Then

$$\varphi(0)=K_{-2}y_{-2}(0)+K_{-1}y_{-1}(0)+K_0y_0(0)+K_1y_1(0)+K_2y_2(0)$$

$$\varphi(-0.5)=K_{-2}y_{-2}(-0.5)+K_{-1}y_{-1}(-0.5)+K_0y_0(-0.5)+K_1y_1(-0.5)+K_2y_2(-0.5)$$
$$\varphi(-1)=K_{-2}y_{-2}(-1)+K_{-1}y_{-1}(-1)+K_0y_0(-1)+K_1y_1(-1)+K_2y_2(-1)$$
$$\varphi(-1.5)=K_{-2}y_{-2}(-1.5)+K_{-1}y_{-1}(-1.5)+K_0y_0(-1.5)+K_1y_1(-1.5)+K_2y_2(-1.5)$$
$$\varphi(-2)=K_{-2}y_{-2}(-2)+K_{-1}y_{-1}(-2)+K_0y_0(-2)+K_1y_1(-2)+K_2y_2(-2)$$

The above system can be written matrix form as follows:

$$\begin{bmatrix}\varphi(0)\\ \varphi(-0.5)\\ \varphi(-1)\\ \varphi(-1.5)\\ \varphi(-2)\end{bmatrix}=\begin{bmatrix} y_{-2}(0) & y_{-1}(0) & y_0(0) & y_1(0) & y_2(0)\\ y_{-2}(-0.5) & y_{-1}(-0.5) & y_0(-0.5) & y_1(-0.5) & y_2(-0.5)\\ y_{-2}(-1) & y_{-1}(-1) & y_0(-1) & y_1(-1) & y_2(-1)\\ y_{-2}(-1.5) & y_{-1}(-1.5) & y_0(-1.5) & y_1(-1.5) & y_2(-1.5)\\ y_{-2}(-2) & y_{-1}(-2) & y_0(-2) & y_1(-2) & y_2(-2)\end{bmatrix}\begin{bmatrix}K_{-2}\\ K_{-1}\\ K_0\\ K_1\\ K_2\end{bmatrix}$$

$$\begin{bmatrix}K_{-2}\\ K_{-1}\\ K_0\\ K_1\\ K_2\end{bmatrix}=\begin{bmatrix} y_{-2}(0) & y_{-1}(0) & y_0(0) & y_1(0) & y_2(0)\\ y_{-2}(-0.5) & y_{-1}(-0.5) & y_0(-0.5) & y_1(-0.5) & y_2(-0.5)\\ y_{-2}(-1) & y_{-1}(-1) & y_0(-1) & y_1(-1) & y_2(-1)\\ y_{-2}(-1.5) & y_{-1}(-1.5) & y_0(-1.5) & y_1(-1.5) & y_2(-1.5)\\ y_{-2}(-2) & y_{-1}(-2) & y_0(-2) & y_1(-2) & y_2(-2)\end{bmatrix}^{-1}\begin{bmatrix}\varphi(0)\\ \varphi(-0.5)\\ \varphi(-1)\\ \varphi(-1.5)\\ \varphi(-2)\end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 4.4232+4.9806i & -1.6424+3.1799i & 0.2126-0.0140i & -3.6249+2.8966i & -5.9154-4.7877i\\ -5.2416+44.0610i & -7.4141-10.4450i & 0.0450-0.0059i & 4.7492-20.9999i & 12.0700+56.6422i\\ -2.4264e+02+1.6879e+02i & 45.3904-6.4213i & 0.0095-0.0019i & 43.6137+89.8789i & 1.9979e+02-3.9285e+02i\\ -1.9139e+03-4.6190e+02i & -5.4129e+01+1.5488e+02i & 0.0020-0.0005i & -4.1844e+02-1.9947e+02i & -3.0627e+03+1.3673e+03i\end{bmatrix}^{-1}\begin{bmatrix}0\\ -0.1271\\ -0.0740\\ -0.0305\\ -0.0108\end{bmatrix}$$

$$=\begin{bmatrix}0.0038-0.0000i\\ 0.0676+0.0325i\\ -0.0379-0.0522i\\ -0.0331+0.0178i\\ -0.0004+0.0018i\end{bmatrix}$$

Therefore the solution of the equation (18) is given as

$$\begin{aligned} y(t) &= (0.0038-0.0000i)e^{(-3.7926-14.2556i)t}+(0.0676+0.0325i)e^{(-2.5501-4.0951i)t}+\\ &(-0.0379-0.0522i)e^{(3.0920+0.1313i)t}+(-0.0331+0.0178i)e^{(-3.0695+7.6316i)t}+\\ &(-0.0004+0.0018i)e^{(-4.0590+17.4887i)t}\end{aligned}$$

The characteristic roots λ for the branches $i=0,\ \pm1,\ \pm2,\ \pm3,$ are as follows:

$i=0;\quad \lambda=\ 3.0920+0.1313i$

$i=1;\quad \lambda=-3.0695+7.6316i$

$i=-1;\quad \lambda=-2.5501-4.0951i$

$i=2;\quad \lambda=-4.0590+17.4887i$

$i=-2;\quad \lambda=-3.7926-14.2556i$

$i=3;\quad \lambda=-4.6591+27.0754i$

$i=-3;\quad \lambda=-4.4839-23.8920i$

From the above characteristic roots we note that $\mathrm{Re}(\lambda)>0$ for some λ then the equation (18) is unstable [3].

Case (2)

$$y(t)=\sum_{i=-\infty}^{\infty}K_i\, e^{[\frac{3}{2}W_i(\frac{-2}{3}e^{-2}\sqrt[3]{2})+3]t},\qquad t\in[-2,0]$$

$$\lambda_i=\frac{3}{2}W_i(\frac{-2}{3}e^{-2}\sqrt[3]{2})+3$$

$$\begin{bmatrix}K_{-2}\\ K_{-1}\\ K_0\\ K_1\\ K_2\end{bmatrix}=\begin{bmatrix} y_{-2}(0) & y_{-1}(0) & y_0(0) & y_1(0) & y_2(0)\\ y_{-2}(-0.5) & y_{-1}(-0.5) & y_0(-0.5) & y_1(-0.5) & y_2(-0.5)\\ y_{-2}(-1) & y_{-1}(-1) & y_0(-1) & y_1(-1) & y_2(-1)\\ y_{-2}(-1.5) & y_{-1}(-1.5) & y_0(-1.5) & y_1(-1.5) & y_2(-1.5)\\ y_{-2}(-2) & y_{-1}(-2) & y_0(-2) & y_1(-2) & y_2(-2)\end{bmatrix}^{-1}\begin{bmatrix}\varphi(0)\\ \varphi(-0.5)\\ \varphi(-1)\\ \varphi(-1.5)\\ \varphi(-2)\end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3.9845-4.0379i & 2.8522 & 0.2459 & 3.9845+4.0379i & -5.1360+6.8047i \\ -0.4289-32.1783i & 8.1348 & 0.0605 & -0.4289+32.1783i & -19.9248-69.8981i \\ -1.3164e+02-1.2648e+02i & 23.2019 & 0.0149 & -1.3164e+02+1.2648e+02i & 5.7797e+02+2.2342e+02i \\ -1.0353e+03+2.7602e+01i & 66.1757 & 0.0037 & -1.0353e+03-2.7602e+01i & -4.4887e+03+2.7854e+03i \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -0.1271 \\ -0.0740 \\ -0.0305 \\ -0.0108 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0064+0.0025\mathrm{i} \\ -0.0415-0.0038\mathrm{i} \\ 0.0498+0.0046\mathrm{i} \\ -0.0029-0.0041\mathrm{i} \\ 0.0009+0.0009\mathrm{i} \end{bmatrix}$$

Therefore the solution of the equation (18) is given as

$$\begin{aligned} y(t) = {} & (-0.0064+0.0025i)e^{(-3.4714-10.9822i)t} + (-0.0415-0.0038i)e^{(-2.0962)t} + (0.0498 \\ & + 0.0046\mathrm{i}\,)e^{(2.8059)t} + (-0.0029-0.0041i)e^{(-3.4714+10.9822i)t} + (0.0009+0.0009i) \\ & .e^{(-4.2861+20.6980i)t} \end{aligned}$$

The characteristic roots λ for the branches $i=0,\ \pm1,\ \pm2,\ \pm3,$ are as follows:

$i=0;\ \ \lambda = 2.8059$

$i=1;\ \ \lambda = -3.4714+10.9822i$

$i=-1;\ \ \lambda = -2.0962$

$i=2;\ \ \lambda = -4.2861+20.6980i$

$i=-2;\ \ \lambda = -3.4714-10.9822i$

$i=3;\ \ \lambda = -4.8162+30.2513i$

$i=-3;\ \ \lambda = -4.2861-20.6980i$

Since, there exists λ such that $\mathrm{Re}(\lambda)>0,$ then the equation (18) is unstable [3]

Case (3)

$$y(t) = \sum_{i=-\infty}^{\infty} K_i\, e^{[\frac{3}{2} W_i(\frac{2}{3} e^{-2}\sqrt[3]{2}(\frac{1}{2}-\frac{\sqrt{3}}{2}i))+3]t}, \quad t \in [-2,0]$$

$$\lambda_i = \frac{3}{2} W_i\left(\frac{2}{3} e^{-2}\sqrt[3]{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)+3$$

$$\begin{bmatrix} K_{-2} \\ K_{-1} \\ K_0 \\ K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} y_{-2}(0) & y_{-1}(0) & y_0(0) & y_1(0) & y_2(0) \\ y_{-2}(-0.5) & y_{-1}(-0.5) & y_0(-0.5) & y_1(-0.5) & y_2(-0.5) \\ y_{-2}(-1) & y_{-1}(-1) & y_0(-1) & y_1(-1) & y_2(-1) \\ y_{-2}(-1.5) & y_{-1}(-1.5) & y_0(-1.5) & y_1(-1.5) & y_2(-1.5) \\ y_{-2}(-2) & y_{-1}(-2) & y_0(-2) & y_1(-2) & y_2(-2) \end{bmatrix}^{-1} \begin{bmatrix} \varphi(0) \\ \varphi(-0.5) \\ \varphi(-1) \\ \varphi(-1.5) \\ \varphi(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -5.9154+4.7877i & -3.6249-2.8966i & 0.2126+0.0140i & -1.6424-3.1799i & 4.4232-4.9806i \\ 12.0700-56.6422i & 4.7492+20.9999i & 0.0450+0.0059i & -7.4141+10.4450i & -5.2416-44.0610i \\ 1.9979e+02+3.9285e+02i & 43.6137-89.8789i & 0.0095+0.0019i & 45.3904+6.4213i & -2.4264e+02-1.6879e+02i \\ -3.0627e+03-1.3673e+03i & -4.1844e+02+1.9947e+02i & 0.0020+0.0005i & -5.4129e+01-1.5488e+02i & -1.9139e+03+4.6190e+02i \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -0.1271 \\ -0.0740 \\ -0.0305 \\ -0.0108 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0000-0.0000i \\ -0.0006+0.0022i \\ 0.0006-0.0017i \\ -0.0000+0.0000i \\ 0.0000-0.0005i \end{bmatrix}$$

Therefore the solution of the equation (18) is given as

$$\begin{aligned} y(t) = {} & (0.0000-0.0000i)e^{(-4.0590-17.4887i)t} + (-0.0006+0.0022i)e^{(-3.0695-7.6316i)t} + \\ & (0.0006-0.0017i)e^{(3.0920-0.1313i)t} + (-0.0000+0.0000i)e^{(-2.5501+4.0951i)t} + \\ & (0.0000-0.0005i)e^{(-3.7926+14.2556i)t} \end{aligned}$$

The characteristic roots λ for the branches $i = 0, \ \pm 1, \ \pm 2, \ \pm 3,$ are as follows:

$i = 0; \ \ \lambda = 3.0920 - 0.1313i$
$i = 1; \ \ \lambda = -2.5501 + 4.0951i$
$i = -1; \ \ \lambda = -3.0695 - 7.6316i$
$i = 2; \ \ \lambda = -3.7926 + 14.2556i$
$i = -2; \ \ \lambda = -4.0590 - 17.4887i$
$i = 3; \ \ \lambda = -4.4839 + 23.8920i$
$i = -3; \ \ \lambda = -4.6591 - 27.0754i$

Since, there exists λ such that $\mathrm{Re}(\lambda) > 0,$ then the equation (18) is unstable [3].

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