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Some Result Zweier I-Convergent triple sequence spaces defined by the double Orlicz functions

Ali Hussein Battor, Elaf Hussein Mohammed

Ali Hussein Battor
Department of Mathematics
Faculty of Education for Girls
University of Kufa, Najaf, Iraq

Elaf Hussein Mohammed
Faculty of Education for Girls
University of Kufa, Najaf, Iraq

Correspondence:
Ali Hussein Battor
Department of Mathematics
Faculty of Education for Girls
University of Kufa, Najaf, Iraq

Abstract
In this paper we introduce the Zweier I-convergent triple sequence spaces $3\mathbb{Z}^{I}(\Psi)$, $3\ \mathbb{Z}_0^{I}(\Psi)$ and $3\mathbb{Z}_{\infty}^{I}(\Psi)$using the double Orlicz functionΨ. We study the algebraic properties and inclusion relations on these spaces.

Keywords: triple sequences, Ideal, double Orlicz function, I-convergent, I-null, solid

1. Introduction
A triple sequence (real or complex) can be defined as a function $T: N \times N \times N \to R(C)$where $N, R\ and\ C$ denote the sets of natural numbers, real numbers and complex numbers respectively [3] [1] .The Orlicz function has been founded by Prof. Wlayshaw Roman Orlicz from Poland and carried his name, so he was constructed the Orlicz space [7].
A double Orlicz function is a function $\Psi: [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty)$such that
$\Psi(k,t) = (\Psi_1(k), \Psi_2(t))$, Where
$\Psi_1 : [0,\infty) \to [0,\infty)\ and\ \Psi_2: [0,\infty) \to [0,\infty)$ such that Ψ_1, Ψ_2 are Orlicz functions which is continuous, non – decreasing, even, convex, and satisfies the following conditions :

1) Ψ_1 (0) =0, Ψ_2 (0) =0 $\Longrightarrow \Psi(0,0) = (\Psi_1\ (0), \Psi_2\ (0)\ ) = (0,0)$
2) (k) > 0, Ψ_2 (t) > 0 $\Longrightarrow\ \Psi(k,t) = (\Psi_1\ (k), \Psi_2\ (t)\ ) > (0,0)$
$for\ k > 0, t > 0\ we\ mean\ by\ \Psi(k,t) > (0,0)\ that\ \Psi_1\ (k) > 0, \Psi_2\ (t) > 0$
3) $\Psi_1\ (k) \to \infty, \Psi_2\ (t) \to \infty$ as $k, t \to \infty$ then,
$\Psi(k,t) = (\Psi_1\ (k), \Psi_2\ (t)\ )\ ) \to (\infty,\infty)\ as\ (k,t) \to (\infty,\infty)$,

We mean by $\Psi(k,t) \to (\infty,\infty)$ that $\Psi_1\ (k) \to \infty$, $\Psi_2(t) \to \infty$.[5][8].
Let X be a non-empty set. Then a family of sets I $\subseteq 2^X$ (power sets of X) is said to be an ideal if I is additive i.e. $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I, B \subseteq A \Rightarrow B \in I$,where X= $sup(k,t)$ [2].

At the initial stage the notion of I-convergence was introduced by Kostyrko, ˇSalat and Wilczyn´ski [4]. Later on it was studied by ˇSalat, Tripathy and Ziman[6], Demirci [9] and many others.
In this paper, we define the Zweier I-convergent triple sequence spaces which is defined by the double Orlicz functions Ψ where$\Psi(k,t) = \left(\Psi_1(k), \Psi_2(t)\right)$, and

Introduce the following classes of Zweier I-convergent triple sequence spaces defined by the double Orlicz functions. Let N, R and C be the sets of all natural, real and complex numbers respectively, we set
$\mu^3 = \left\{(\mathrm{k},\mathrm{t}) = \left(\mathrm{k}_{\mathrm{h,d,b}}, \mathrm{t}_{\mathrm{h,d,b}}\right) : \left(\mathrm{k}_{\mathrm{h,d,b}}, \mathrm{t}_{\mathrm{h,d,b}}\right) \in\ \mathrm{R} \times \mathrm{R}\ \times \mathrm{R}\ \text{or}\ \mathrm{C} \times \mathrm{C} \times \mathrm{C}\right\}$

The space of all triple sequences real or complex .Throughout this work the triple sequence will be denoted by$(\mathrm{k},\mathrm{t}) = \left(\mathrm{k}_{\mathrm{h,d,b}}, \mathrm{t}_{\mathrm{h,d,b}}\right)$ i.e., a triple infinite array of elements $\left(\mathrm{k}_{\mathrm{h,d,b}}, \mathrm{t}_{\mathrm{h,d,b}}\right)$ for all $h, d, b \in \mathbb{N}$, we mean that $\mathrm{k} = \left(\mathrm{k}_{\mathrm{h,d,b}}\right), \mathrm{t} = \left(\mathrm{t}_{\mathrm{h,d,b}}\right)$

be an infinite array of elements for all $h,d,b \in \mathbb{N}$.We define the Banach spaces of I- bounded, I-convergent, I-null, bounded I-convergent and bounded I-null triple sequences normed by $\|(k,t)\|_\infty = sup_{h,d,b} |(\mathrm{k_{h,d,b}}, \mathrm{t_{h,d,b}})|$ And we study their different properties like soild, symmetricity, monotony etc .

2. Some Important Preliminaries and Concepts

An Zweier triple sequence spaces $\mathbb{Z}^3$, $\mathbb{Z}_0^3$ and $\mathbb{Z}_\infty^3$ are defined as follows:

$\mathbb{Z}^3 = \{(k,t) = (k_x, t_x) \in \mu^3 : (Z^3)^p(k,t) \in (C^3)^I$, where $x = h,d,b\}$,

$\mathbb{Z}_0^3 = \{(k,t) = (k_x, t_x) \in \mu^3 : (Z^3)^p (k,t) \in (C_0^3)^I$, where $x = h,d,b\}$,

$\mathbb{Z}_\infty^3 = \{(k,t) = (k_x, t_x) \in \mu^3 : (Z^3)^p (k,t) \in (C_\infty^3)^I$, where $x = h,d,b\}$,where $(Z^3)^p$denoted the matrix $(Z^3)^p = (z_{ix})$ defined by

$$z_{ix} = \begin{cases} p, (i = x) \\ 1-p, (i-1 = x); i,x \in N \\ 0, otherwise \end{cases}$$

Now, we introduced the following classes of triple sequence spaces.

$(\mathbb{Z}^3)^I = \{h,d,b \in N : \{(k,t) = (k_{h,d,b}, t_{h,d,b}) \in \mu^3 :$ I $-\lim (Z^3)^p(k,t) = (\ell_1,\ell_2)$ $for\ some\ \ell_1,\ell_2\}\} \in I$, where

I $-\lim (Z^3)^p k = \ell_1$ $for\ some\ \ell_1$,I $-\lim (Z^3)^p t = \ell_2$ $for\ some\ \ell_2$

$(\mathbb{Z}_0^3)^I = \{h,d,b \in N : \{(k,t) = (k_{h,d,b}, t_{h,d,b}) \in \mu^3 :$ I $-\lim (Z^3)^p(k,t) = (0,0)\} \in I$, where

I $-\lim (Z^3)^p k = 0$, I $-\lim (Z^3)^p t = 0$

$(\mathbb{Z}_\infty^3)^I = \{h,d,b \in N : \{(k,t) = (k_{h,d,b}, t_{h,d,b}) \in \mu^3 : sup_{h,d,b} |(Z^3)^p (k,t)| < (\infty,\infty)\}\} \in$ I, where

$sup_{h,d,b} |(Z^3)^p k| < \infty$, $sup_{h,d,b} |(Z^3)^p t| < \infty$.

We also denote by

$\left(m_{\mathbb{Z}^3}^3\right)^I = (\mathbb{Z}_\infty^3)^I \cap (\mathbb{Z}^3)^I$ and $\left(m_{\mathbb{Z}_0^3}^3\right)^I = (\mathbb{Z}_\infty^3)^I \cap (\mathbb{Z}_0^3)^I$

In this section we introduce the following classes of Zweier I-Convergent triple sequence spaces defined by the double Orlicz functions.

$3\mathbb{Z}^I(\Psi) = \{(k,t) = (k_{h,d,b}, t_{h,d,b}) \in \mu^3 :$

$$I - lim\left[\sup\left\{\left(\Psi_1\left(\frac{|\mathrm{k'_{h,d,b}} - \ell_1|}{\rho}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{t'_{h,d,b}} - \ell_2|}{\rho}\right)\right)\right\}\right] = 0, for\ some\ \ell_1, \ell_2\ and\ \rho > 0\},$$

$3\ \mathbb{Z}_0^I(\Psi) = \{(k,t) = (k_{h,d,b}, t_{h,d,b}) \in \mu^3 :$

$$I - lim\left[\sup\left\{\left(\Psi_1\left(\frac{|\mathrm{k'_{h,d,b}}|}{\rho}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{t'_{h,d,b}}|}{\rho}\right)\right)\right\}\right] = 0, for\ some\ \rho > 0\},$$

$3\mathbb{Z}_\infty^I(\Psi) = \{(k,t) = (k_{h,d,b}, t_{h,d,b}) \in \mu^3 :$

$$sup_{h,d,b}\left[\sup\left\{\left(\Psi_1\left(\frac{|\mathrm{k'_{h,d,b}}|}{\rho}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{t'_{h,d,b}}|}{\rho}\right)\right)\right\}\right] < \infty, for\ some\ \rho > 0\}.$$

Also we denoted by

$3m_{\mathbb{Z}^3}^I(\Psi) = 3\mathbb{Z}_\infty^I(\Psi) \cap 3\mathbb{Z}^I(\Psi)$ and $3\ m_{\mathbb{Z}_0^3}^I(\Psi) = 3\mathbb{Z}_\infty^I(\Psi) \cap 3\mathbb{Z}_0^I(\Psi)$.

We will denoteby$(Z^3)^p(k_{h,d,b}, t_{h,d,b}) = (\mathrm{k'_{h,d,b}}, \mathrm{t'_{h,d,b}})$,where $(Z^3)^p(k_{h,d,b}) = (\mathrm{k'_{h,d,b}})$ and $(Z^3)^p(t_{h,d,b}) = (\mathrm{t'_{h,d,b}})$, $(Z^3)^p(a_{h,d,b}, j_{h,d,b}) = (\mathrm{a'_{h,d,b}}, \mathrm{j'_{h,d,b}})$, where $(Z^3)^p(a_{h,d,b}) = (\mathrm{a'_{h,d,b}})$ and $(Z^3)^p(j_{h,d,b}) = (\mathrm{j'_{h,d,b}})$.

Lemma .2.1

[5].A sequence space E is solid implies that E is monotone.

Definition 2.1

A triple sequence space $(\mathrm{k},\mathrm{t}) \in \mu 3$ is said to be I-convergent to the number (ℓ_1,ℓ_2) if for every ϵ>0,

$\{h,d,b \in N : |(k_{h,d,b} - \ell_1, t_{h,d,b} - \ell_2)| \geq \epsilon\} \in I$. In this case we write

$I - lim_{h,d,b}(k_{h,d,b}, t_{h,d,b}) = (\ell_1,\ell_2)$, where

$I - lim_{h,d,b} k_{h,d,b} = \ell_1$,

$I - lim_{h,d,b} t_{h,d,b} = \ell_2$.

Definition 2.2

A triple sequence $(\mathrm{k},\mathrm{t}) \in \mu 3$ is said to be I-null if $(\ell_1,\ell_2) =$ (0,0), where $\ell_1 = 0, \ell_2 = 0$.

In this case we write$I - lim_{h,d,b}(k_{h,d,b}, t_{h,d,b}) = (0,0)$.

Definition 2.3

A triple sequence $(\mathrm{k},\mathrm{t}) \in \mu 3$ is said to be I-Cauchy if for every ϵ >0 there exists a number $n,r,s = n,r,s(\epsilon)$ such that

$\{h,d,b \in N : |(k_{h,d,b}, t_{h,d,b}) - (k_{n,r,s}, t_{n,r,s})| \geq \epsilon\} \in I$.

Definition 2.4

A triple sequence$(\mathrm{k},\mathrm{t}) \in \mu 3$ is said to be I-bounded if there exists Y>0 such that $\{h,d,b \in N : |(k_{h,d,b}, t_{h,d,b})| >Y\} \in I$.

Remark .2.1 [3]

If Ψ is an Orlicz function, then $\Psi(\lambda k) \leq \lambda\Psi(k)$ for all λ with $0 < \lambda < 1$.

Lemma .2.2

That a triple sequence spaceE^3 is solid implies that E^3 is monotone.

Definition 2.5

A triple sequence space E^3 is said to be solid (or normal) if $(\alpha_{h,d,b}k_{h,d,b}, \beta_{h,d,b}t_{h,d,b}) \in E^3$ whenever $(k_{h,d,b}, t_{h,d,b}) \in E^3$and for all triple sequence $(\alpha_{h,d,b}, \beta_{h,d,b})$ of scalars with $|\alpha_{h,d,b}| \leq 1, |\beta_{h,d,b}| \leq 1$ for all $h,d,b \in N$.

Definition 2.6

A triple sequence space $E^3$is said to be symmetric if $(k_{h,d,b}, t_{h,d,b}) \in E^3$implies $(k_{\pi(h),\pi(d),\pi(b)}, t_{\pi(h),\pi(d),\pi(b)}) \in E^3$, where π is a permutation of N.

Definition 2.7

A triple sequence space E^3 is said to be sequence algebra if $(k_{h,d,b}) * (t_{h,d,b}) = (k_{h,d,b}t_{h,d,b}) \in$ E whenever $(k_{h,d,b}), (t_{h,d,b}) \in E^3$.

Definition 2.8

A triple sequence space $E^3$is said to be convergence free if $(a_{h,d,b}, j_{h,d,b}) \in E^3$whenever $(k_{h,d,b}, t_{h,d,b}) \in E^3$and $(k_{h,d,b}, t_{h,d,b})$ =0 implies $(a_{h,d,b}, j_{h,d,b})$ = 0.

Definition 2.9

Let $H = \{h_1 < h_2 < \cdots\} \subset N$, $D = \{d_1 < d_2 < \cdots\} \subset$

N and
$B = \{b_1 < b_2 < \cdots\} \subset N$, Let E^3 be a triple sequence space. A (H, D, B)-step space of E^3 is a triple sequence space $\tau^{E^3}_{H,D,B} = \{((k_{h_n,d_r,b_s}, t_{h_n,d_r,b_s})) \in \mu^3 : (k_{n,r,s}, t_{n,r,s}) \in E^3\}$.

Definition 2.10
A canonical preimage of a sequence $(k_{h_n,d_r,b_s}, t_{h_n,d_r,b_s}) \in \tau^{E^3}_{H,D,B}$ is a triple sequence $(a_{h,d,b}, j_{h,d,b}) \in E^3$ defined by

$$(a_{h,d,b}, j_{h,d,b}) = \begin{cases} (k_{n,r,s}, t_{n,r,s}) & if\ n,r,s \in N \\ 0 & otherwise \end{cases}$$

Definition 2.11
A canonical preimage of a step space $\tau^{E^3}_{H,D,B}$ is a set of canonical preimages of all elements in $\tau^{E^3}_{H,D,B}$, i.e. (a,j) is in the canonical preimage of $\tau^{E^3}_{H,D,B}$ if and only if (a,j) is a canonical preimage of some $(k,t) \in \tau^{E^3}_{H,D,B}$.

Definition 2.12
A triple sequence space E^3 is said to be monotone if it contains the canonical preimages of its step spaces.
Now, we introduce the following classes of triple sequences spaces:

$$T^3_\infty(\Psi) = \left\{ (\mathrm{k},\mathrm{t}) \in \mu 3 : sup_{h,d,b} \sup \left\{ \Psi_1\left(\frac{|\mathrm{k}_{h,d,b}|}{\rho}\right), \Psi_2\left(\frac{|t_{h,d,b}|}{\rho}\right) \right\} < \infty, \text{for some} \rho > 0 \right\},$$
i.e $T^3_\infty(\Psi) = (3T_\infty(\Psi_1), 3T_\infty(\Psi_2))$

$$(C^3)^I(\Psi) = \left\{ (\mathrm{k},\mathrm{t}) \in \mu^3 : \mathrm{I} - \lim \left[\sup \left\{ \left(\Psi_1\left(\frac{|\mathrm{k}_{h,d,b} - \ell_1|}{\rho}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{t}_{h,d,b} - \ell_2|}{\rho}\right)\right) \right\} \right] = 0, \text{ for some } \ell_1, \ell_2 \text{ and } \rho > 0 \right\},$$
i.e $(C^3)^I(\Psi) = (3C^I(\Psi_1), 3C^I(\Psi_1))$

$$(C^3_0)^I(\Psi) = \left\{ (\mathrm{k},\mathrm{t}) \in \mu^3 : \mathrm{I} - \lim \left[\sup \left\{ \left(\Psi_1\left(\frac{|\mathrm{k}_{h,d,b}|}{\rho}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{t}_{h,d,b}|}{\rho}\right)\right) \right\} \right] = 0, \text{ for some } \rho > 0 \right\},$$
i.e $(C^3_0)^I(\Psi) = (C^I_o(\Psi_1), C^I_o(\Psi_2))$.
Furthermore, we write
$(m^3)^I(\Psi) = (C^3)^I(\Psi) \cap T^3_\infty(\Psi)$ and $(m^3_0)^I(\Psi) = (C^3_0)^I(\Psi) \cap T^3_\infty$.

3. Main Results
Theorem 3.2.2.1 for any double Orlicz function Ψ, the classes of triple sequences $3\mathbb{Z}^I(\Psi)$, $3\mathbb{Z}^I_0(\Psi)$, $3m^I_{\mathbb{Z}^3}(\Psi)$ and $3m^I_{\mathbb{Z}^3_0}(\Psi)$ are linear spaces.
Proof. We will prove the result for the space $3\mathbb{Z}^I(\Psi)$ .The proof for the other spaces will follow similarly. Let $(k_{h,d,b}), (a_{h,d,b}) \in 3\mathbb{Z}^I(\Psi_1)$ and $(t_{h,d,b}), (j_{h,d,b}) \in 3\mathbb{Z}^I(\Psi_2)$ and consequently $(k,t) = (k_{h,d,b}, t_{h,d,b}) \in (\mathbb{Z}^3)^I(\Psi)$, $(a,j) = (a_{h,d,b}, j_{h,d,b}) \in (\mathbb{Z}^3)^I(\Psi)$ and let (α,α), (β, β) be scalars. Then there exists positive numbers ρ_1 and ρ_2 such that

$$\mathrm{I}-\lim\left[\sup\left\{\left(\Psi_1\left(\frac{|\mathrm{k}'_{h,d,b}-\ell_1|}{\rho_1}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{t}'_{h,d,b}-\ell_1|}{\rho_1}\right)\right)\right\}\right] = 0, \text{ for some } \ell_1, \ell_2 \in \mathbb{C};$$

$$\mathrm{I}-\lim\left[\sup\left\{\left(\Psi_1\left(\frac{|\mathrm{a}'_{h,d,b}-\ell_2|}{\rho_2}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{j}'_{h,d,b}-\ell_2|}{\rho_2}\right)\right)\right\}\right] = 0, \text{ for some } \ell_1, \ell_2 \in \mathbb{C}.$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \left\{ h,d,b \in \mathrm{N} : \left[\sup \left\{ \left(\Psi_1\left(\frac{|\mathrm{k}'_{h,d,b} - \ell_1|}{\rho_1}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{t}'_{h,d,b} - \ell_1|}{\rho_1}\right)\right) \right\} \right] > \frac{\epsilon}{2} \right\} \in I, \quad (1)$$

$$A_2 = \left\{ h,d,b \in N : \left[\sup \left\{ \left(\Psi_1\left(\frac{|\mathrm{a}'_{h,d,b} - \ell_2|}{\rho_2}\right)\right), \left(\Psi_2\left(\frac{|\mathrm{j}'_{h,d,b} - \ell_2|}{\rho_2}\right)\right) \right\} \right] > \frac{\epsilon}{2} \right\} \in I, \quad (2)$$

Let $\rho_3 = \max\{3|\alpha|\rho_1, 3|\beta|\rho_2\}$. Since Ψ_1, Ψ_2 and Ψ are non – decreasing and convex functions,we have

$$\sup\left\{\left(\Psi_1\left(\frac{|[\alpha(\mathrm{k}'_{h,d,b}) + \beta(\mathrm{a}'_{h,d,b})] - [\alpha\ell_1 + \beta\ell_2]|}{\rho_3}\right)\right), \left(\Psi_2\left(\frac{|[\alpha(\mathrm{t}'_{h,d,b}) + \beta(\mathrm{j}'_{h,d,b})] - [\alpha\ell_1 + \beta\ell_2]|}{\rho_3}\right)\right)\right\} \leq$$

$$\sup\left\{\left(\Psi_1\left(\frac{|\alpha|\left|k'_{h,d,b}-\ell_1\right|}{\rho_3}+\frac{|\beta|\left|a'_{h,d,b}-\ell_2\right|}{\rho_3}\right)\right),\left(\Psi_2\left(\frac{|\alpha|\left|t'_{h,d,b}-\ell_1\right|}{\rho_3}+\frac{|\beta|\left|j'_{h,d,b}-\ell_2\right|}{\rho_3}\right)\right)\right\}\leq$$

$$\sup\left\{\left(\Psi_1\left(\frac{\left|k'_{h,d,b}-\ell_1\right|}{\rho_1}+\frac{\left|a'_{h,d,b}-\ell_2\right|}{\rho_2}\right)\right),\left(\Psi_2\left(\frac{\left|t'_{h,d,b}-\ell_1\right|}{\rho_1}+\frac{\left|j'_{h,d,b}-\ell_2\right|}{\rho_2}\right)\right)\right\}$$

Now, by (1) and (2), we have

$$\left\{h,d,b\in N:\sup\left\{\left(\Psi_1\left(\frac{|[\alpha(k'_{h,d,b})+\beta(a'_{h,d,b})]-[\alpha\ell_1+\beta\ell_2]|}{\rho_3}\right)\right),\left(\Psi_2\left(\frac{|[\alpha(t'_{h,d,b})+\beta(j'_{h,d,b})]-[\alpha\ell_1+\beta\ell_2]|}{\rho_3}\right)\right)\right\}>\epsilon\right\}\subset A_1\cup A_2.$$

Therefore $[\alpha(k_{h,d,b},t_{h,d,b})+\beta(a_{h,d,b},j_{h,d,b})]\in 3\mathbb{Z}^I(\Psi)$. Then $3\mathbb{Z}^I(\Psi)$ is a linear space.

Theorem 3.2.2.2

Let Ψ_1,Ψ_2 be a double Orlicz functions that satisfy the Δ_2-condition. Then

(i) $\sigma^3(\Psi_2)\subseteq\sigma^3(\Psi_1.\Psi_2)$.

(ii) $\sigma^3(\Psi_1)\cap\sigma^3(\Psi_2)\subseteq\sigma^3(\Psi_1+\Psi_2)$ for $\sigma^3=3\mathbb{Z}^I(\Psi)$, $3\mathbb{Z}_0^I(\Psi)$, $3m_{\mathbb{Z}^3}^{I}(\Psi)$ and $3m_{\mathbb{Z}_0^3}^I(\Psi)$, where $\Psi_1=(\Psi_3,\Psi_4)$ and $\Psi_2=(\Psi_5,\Psi_6)$.

Proof. (i) Let $(k_{h,d,b},t_{h,d,b})\in 3\mathbb{Z}_0^I(\Psi_2)$, where $k_{h,d,b}\in 3\mathbb{Z}_0^I(\Psi_5)$ and $t_{h,d,b}\in 3\mathbb{Z}_0^I(\Psi_6)$. Then there exists ρ>0 such that

$$I-lim_{h,d,b}sup\left\{\Psi_5\left(\frac{((k'_{h,d,b}))}{\rho}\right),\Psi_6\left(\frac{((t'_{h,d,b}))}{\rho}\right)\right\}=0 \tag{3}$$

Let ϵ>0 and choose δ with 0 <δ<1 such that Ψ_1 (v) $<\epsilon$ for 0≤ v ≤ δ. We write $(a_{h,d,b},j_{h,d,b})=sup\left\{\Psi_5\left(\frac{((k'_{h,d,b}))}{\rho}\right),\Psi_6\left(\frac{(t'_{h,d,b})}{\rho}\right)\right\}$ and consider for all $(h,d,b)\in N$, we have

$$lim_{0\leq(a_{h,d,b},j_{h,d,b})\leq\delta,h,d,b,\in N}sup\left\{\Psi_3\left(\frac{(a_{h,d,b})}{\rho}\right),\Psi_4\left(\frac{(j_{h,d,b})}{\rho}\right)\right\}=$$
$$lim_{(a_{h,d,b},j_{h,d,b})\leq\delta,h,d,b,\in N}sup\left\{\Psi_3\left(\frac{(a_{h,d,b})}{\rho}\right),\Psi_4\left(\frac{(j_{h,d,b})}{\rho}\right)\right\}+$$
$$lim_{(a_{h,d,b},j_{h,d,b})>\delta,h,d,b,\in N}sup\left\{\Psi_3\left(\frac{(a_{h,d,b})}{\rho}\right),\Psi_4\left(\frac{(j_{h,d,b})}{\rho}\right)\right\}.$$

We have

$$lim_{(a_{h,d,b},j_{h,d,b})\leq\delta,h,d,b,\in N}sup\left\{\Psi_3\left(\frac{(a_{h,d,b})}{\rho}\right),\Psi_4\left(\frac{(j_{h,d,b})}{\rho}\right)\right\}\leq\Psi_1(2)lim_{(a_{h,d,b},j_{h,d,b})\leq\delta,h,d,b,\in N}(a_{h,d,b},j_{h,d,b}). \tag{4}$$

For $(a_{h,d,b},j_{h,d,b})$ >δ, we have $(a_{h,d,b},j_{h,d,b})<\frac{(a_{h,d,b},j_{h,d,b})}{\delta}<1+\frac{(a_{h,d,b},j_{h,d,b})}{\delta}$, where $(a_{h,d,b})<\frac{(a_{h,d,b})}{\delta}<1+\frac{(a_{h,d,b})}{\delta}$, $(j_{h,d,b})<\frac{(j_{h,d,b})}{\delta}<1+\frac{(j_{h,d,b})}{\delta}$.

Since Ψ_1 is non-decreasing and convex, it follows that

$\Psi_1(a_{h,d,b},j_{h,d,b})<\Psi_1\left(1+\frac{(a_{h,d,b},j_{h,d,b})}{\delta}\right)<\frac{1}{2}\Psi_1(2)+\frac{1}{2}\Psi_1\left(\frac{2(a_{h,d,b},j_{h,d,b})}{\delta}\right)$,

where $\Psi_1(a_{h,d,b},j_{h,d,b})=sup\left\{\Psi_3\left(\frac{(a_{h,d,b})}{\rho}\right),\Psi_4\left(\frac{(j_{h,d,b})}{\rho}\right)\right\}$.

Since Ψ_1 satisfies the Δ_2-condition, we have

$\Psi_1(a_{h,d,b},j_{h,d,b})<\frac{1}{2}Y\frac{(a_{h,d,b},j_{h,d,b})}{\delta}\Psi_1(2)+\frac{1}{2}Y\frac{(a_{h,d,b},j_{h,d,b})}{\delta}\Psi_1(2)=Y\frac{(a_{h,d,b},j_{h,d,b})}{\delta}\Psi_1(2)$.

Hence

$$lim_{(a_{h,d,b},j_{h,d,b})>\delta,h,d,b,\in N}sup\left\{\Psi_3\left(\frac{(a_{h,d,b})}{\rho}\right),\Psi_4\left(\frac{(j_{h,d,b})}{\rho}\right)\right\}\leq\max(1,Y\delta^{-1}\Psi_1(2))\,lim_{(a_{h,d,b},j_{h,d,b})>\delta,h,d,b,\in N}(a_{h,d,b},j_{h,d,b}). \tag{5}$$

From (3), (4) and (5), we have $(k_{h,d,b},t_{h,d,b})\in 3\mathbb{Z}_0^I(\Psi_1.\Psi_2)$.

Thus $3\mathbb{Z}_0^I(\Psi_2)\subseteq 3\mathbb{Z}_0^I(\Psi_1\cdot\Psi_2)$. The other cases can be proved similarly.

(ii) Let $(k_{h,d,b},t_{h,d,b})\in\mathbb{Z}_0^I(\Psi_1)\cap\mathbb{Z}_0^I(\Psi_2)$.

Then there exists ρ_1>0 and ρ_2>0 such that

$I-lim_{h,d,b}sup\left\{\Psi_3\left(\frac{(k_{h,d,b})}{\rho_1}\right),\Psi_4\left(\frac{(t_{h,d,b})}{\rho_1}\right)\right\}=0$, and

$I-lim_{h,d,b}sup\left\{\Psi_5\left(\frac{(k_{h,d,b})}{\rho_2}\right),\Psi_6\left(\frac{(t_{h,d,b})}{\rho_2}\right)\right\}=0$

Let $\rho=max\{\rho_1,\rho_2\}$, the rest of the proof follows from the following equality

$$lim_{h,d,b}sup\left\{(\Psi_3+\Psi_5)\left(\frac{(k_{h,d,b})}{\rho}\right),(\Psi_4+\Psi_6)\left(\frac{(t_{h,d,b})}{\rho}\right)\right\}=$$
$$lim_{h,d,b}sup\left\{(\Psi_3)\left(\frac{|(k_{h,d,b})|}{\rho_1}\right)+(\Psi_5)\left(\frac{|(k_{h,d,b})|}{\rho_2}\right),(\Psi_4)\left(\frac{|(t_{h,d,b})|}{\rho_1}\right)+(\Psi_6)\left(\frac{|(t_{h,d,b})|}{\rho_2}\right)\right\}$$

Theorem .3.2.2.1

The spaces $3\mathbb{Z}_0^I(\Psi)$ and $3m_{\mathbb{Z}_0^3}^I(\Psi)$ are solid and monotone

Proof. We shall prove the result for $3\mathbb{Z}_0^I(\Psi)$. for $3m_{\mathbb{Z}_0^3}^I(\Psi)$, the result can be proved similarly. Let $(k_{h,d,b},t_{h,d,b})\in 3\mathbb{Z}_0^I(\Psi)$, where $k_{h,d,b}\in 3\mathbb{Z}_0^I(\Psi_1)$ and $t_{h,d,b}\in 3\mathbb{Z}_0^I(\Psi_2)$. Then there exists ρ>0 such that

$$I-lim_{h,d,b}\left[\sup\left\{\left(\Psi_1\left(\frac{|(k'_{h,d,b})|}{\rho}\right)\right),\left(\Psi_2\left(\frac{|(t'_{h,d,b})|}{\rho}\right)\right)\right\}\right]=0. \tag{6}$$

Let $(\alpha_{h,d,b})$ be a sequence of scalars with $|\alpha_{h,d,b}|\leq 1$ for all $h,d,b\in N$.

Then the rest follows from (6) and the following inequality

$$\left[\sup\left\{\left(\Psi_1\left(\frac{|\alpha_{h,d,b}(k'_{h,d,b})|}{\rho}\right)\right),\left(\Psi_2\left(\frac{|\alpha_{h,d,b}(t'_{h,d,b})|}{\rho}\right)\right)\right\}\right]\leq|\alpha_{h,d,b}|\left[\sup\left\{\left(\Psi_1\left(\frac{k'_{h,d,b}}{\rho}\right)\right),\left(\Psi_2\left(\frac{|t'_{h,d,b}|}{\rho}\right)\right)\right\}\right]\leq$$
$$\left[\sup\left\{\left(\Psi_1\left(\frac{|k'_{h,d,b}|}{\rho}\right)\right),\left(\Psi_2\left(\frac{|t'_{h,d,b}|}{\rho}\right)\right)\right\}\right].$$

By lemma.3.2.1.1 the triple sequence space E^3 is solid implies that E^3 is monmtone .We have the space $3\mathbb{Z}_0^I(\Psi)$ is monotone.

Theorem 3.2.2.2

The triple spaces $3\mathbb{Z}^I(\Psi)$ and $3m_{\mathbb{Z}^3}^I$ $(\Psi)$are neither solid nor monotone in general.

Proof. The proof of this result follows from the following example.

Example 3.2.2.1

Let $I = I_\delta$, $\Psi_1(k)=k^2$ and $\Psi_2\ (t)=t^2$ for all $k,t\ \in\ [0,\infty)$. Consider the (H,D,B)-step space $U_{H,D,B}(\Psi)$ of U(Ψ) defined as follows:

Let$(k_{h,d,b}, t_{h,d,b}) \in$ U(Ψ) and let $(a_{h,d,b}, j_{h,d,b}) \in U_{H,D,B}(\Psi)$ be such that

$$(a_{h,d,b}, j_{h,d,b}) = \begin{cases} (k_{h,d,b}, t_{h,d,b}), if\,(h+d+b)\ is\ even \\ (0,0)\ otherwise \end{cases}$$

Consider the sequence $(k_{h,d,b}, t_{h,d,b})$ defined by $(k_{h,d,b}, t_{h,d,b})$= (1, 1) for all $h,d,b\ \in\ N$ Then $(k_{h,d,b}, t_{h,d,b})\ \in\ 3\mathbb{Z}^I(\Psi)$ but its (H,D,B)-step space preimage does not belong to $3\mathbb{Z}^I(\Psi)$. Thus $3\mathbb{Z}^I(\Psi)$is not monotone. Hence $3\mathbb{Z}^I(\Psi)$ is not solid.

Theorem 3.2.2.3

The spaces $3\ \mathbb{Z}_0^I(\Psi)$ and$3\mathbb{Z}^I(\Psi)$are not convergence free in general.

Proof. The proof of this result follows from the following example.

Example 3.2.2.2

Let $I = I_\gamma$, $\Psi_1(k)=k^3$ and $\Psi_2\ (t)=t^3$ for all $k,t\ \in\ [0,\infty)$. Consider the sequences $(k_{h,d,b}, t_{h,d,b})$ and $(a_{h,d,b}, j_{h,d,b})$ defined by

$(k_{h,d,b}, t_{h,d,b}) = \left(\frac{1}{h+d+b}, \frac{1}{h+d+b}\right)$ and $(a_{h,d,b}, j_{h,d,b}) = (h+d+b, h+d+b)$ for all$h,d,b\ \in N$. Then $(k_{h,d,b}, t_{h,d,b}) \in 3\ \mathbb{Z}_0^I(\Psi)$ and$3\mathbb{Z}^I(\Psi)$, but $(a_{h,d,b}, j_{h,d,b}) \notin 3\ \mathbb{Z}_0^I(\Psi)$ and $3\mathbb{Z}^I(\Psi)$. Hence the spaces 3 $\mathbb{Z}_0^I(\Psi)$ and $3\mathbb{Z}^I(\Psi)$ are not convergence free.

Theorem 3.2.2.4. The spaces $3\ \mathbb{Z}_0^I(\Psi)$ and $3\mathbb{Z}^I(\Psi)$ are sequence algebras.

Proof. We prove that $3\ \mathbb{Z}_0^I(\Psi)$ is sequence algebra. For the space$3\mathbb{Z}^I(\Psi)$, the result can be proved similarly.

Let$(k_{h,d,b}, t_{h,d,b})$, $(a_{h,d,b}, j_{h,d,b}) \in 3\ \mathbb{Z}_0^I(\Psi)$,then

$$\mathrm{I}-lim_{\ h,d,b}\left[\sup\left\{\left(\Psi_1\ \left(\frac{|k'_{h,d,b}|}{\rho_1}\right)\right)\ ,\left(\Psi_2\ \left(\frac{|t'_{h,d,b}|}{\rho_1}\right)\right)\right\}\right] = 0 \text{ for some } \rho_1 > 0, \text{ and}$$

$$\mathrm{I}-lim_{\ h,d,b}\left[\sup\left\{\left(\Psi_1\ \left(\frac{|a'_{h,d,b}|}{\rho_2}\right)\right)\ ,\left(\Psi_2\ \left(\frac{|j'_{h,d,b}|}{\rho_2}\right)\right)\right\}\right] = 0 \text{ for some} \rho_2 > 0, \text{ where} \Psi = (\Psi_1, \Psi_2).$$

Let $\rho = \rho_1 . \rho_2 > 0$. Then we can show that,

$$\mathrm{I}-lim_{\ h,d,b}\left[\sup\left\{\left(\Psi_1\ \left(\frac{|(k'_{h,d,b}\cdot a'_{h,d,b})|}{\rho}\right)\right)\ ,\left(\Psi_2\ \left(\frac{|(t'_{h,d,b}\cdot j'_{h,d,b})|}{\rho}\right)\right)\right\}\right] = 0.$$

Thus $\left[(k_{h,d,b}\ \cdot\ a_{h,d,b}), (t_{h,d,b}\ \cdot\ j_{h,d,b})\right] \in 3\ \mathbb{Z}_0^I(\Psi)$.

Hence $3\ \mathbb{Z}_0^I(\Psi)$ is sequence algebra.

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