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Bayesian Estimation for ISB p-dim Rayleigh distribution under type-II censored data using Lindley's Approximation

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Abstract
In this paper, Bayesian estimation of the unknown parameter of inverse size biased (ISB) p-dimensional (p-dim) Rayleigh distribution under Type-II censored samples is considered. The Bayes estimate of the parameter is obtained under squared error, LINEX and general entropy loss functions with natural conjugate prior by Lindley's approximation method. Further, the comparison of Bayes estimators with corresponding maximum likelihood estimators have been carried out through simulation study.

Keywords: ISB p-dim Rayleigh distribution, Type-II censoring scheme, Lindley's approximation

1. Introduction
In this paper, the Rayleigh distribution is considered as a useful life time distribution. It plays an important role in statistics and operations research. Rayleigh model is applied in several areas such as health, agriculture, biology and physics. It is often used in physics, to model processes such as sound and light radiation, wave heights, as well as in communication theory to describe hourly median and instantaneous peak power of received radio signals. The model for frequency of different wind speeds over a year at wind turbine sites and daily average wind speed are considered under the Rayleigh model. The Rayleigh distribution was introduced by Lord Rayleigh in1980; and used in reliability theory and survival analysis because of its simplicity. Soliman (2000) obtained the Bayes estimators for the parameter, the reliability function, and failure-rate function of the Rayleigh distribution based on complete or type-II censored samples.
The p-dimensional Rayleigh distribution was introduced by Cohen and Whitten (1988). The ISB p-dim Rayleigh distribution was introduced by Pandey and Kumari (2016) under different loss functions with Hartigan prior for Bayesian estimation with complete data set. The ISB p-dim Rayleigh distribution is a new life time distribution. The probability and cumulative density functions of the ISB p-dim Rayleigh distribution is defined as follows

$$f(y;\alpha)=\frac{2}{\alpha^{(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)}\frac{1}{y^{p+2}}\exp\left(-\frac{1}{\alpha y^2}\right);\ 0<y<\infty \qquad (1.1)$$

$$F(t)=\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\Gamma\left(\frac{p+1}{2},\frac{1}{\alpha t^2}\right);t>0 \qquad (1.2)$$

Where, p is a positive known quantity and α>0. This is upper incomplete gamma function.
Censoring, defined as the loss of observations on the lifetime variable of interest, in theprocess of investigation, may occur in life testing experiment due to lack of time, scarcity of funds or any other unavoidable reasons. In type-II censoring scheme some units (say, *n*)

are placed on test and the test is terminated after observing the lifetime of a prefixed number, say *m*($\le$ *n*) units. Thus out of *n*, the lifetimes of *m* units are observed and *n*-*m* units are considered as censored. The prior distribution for the parameter of the model has been taken as a natural conjugate prior. The squared error, LINEX and general entropy loss functions for parameter α is given by

$$L(\theta,\hat{\theta})=(\theta-\hat{\theta})^2\,,\; L(\theta-\hat{\theta})=b\left(e^{k(\theta-\hat{\theta})}-k(\theta-\hat{\theta})-1\right);b>0,k\neq 0$$

$$L(\delta)=\left(\delta^q-q\log_e(\delta)-1\right);\;\delta=\frac{\hat{\alpha}}{\alpha}\; and\;\; q=1$$

where, *b* and *k* are shape and scale parameters of the loss function. The squared-error loss function (SELF) is a symmetric function such that the losses due to overestimation and underestimation of equivalent extent are regarded as equally risky. Nonetheless, such a limitation is not always equally realistic. Square error loss function is seen as a specific case of Linear Exponential Loss Function (LINEX) loss function for an individual choice of the loss function. Asymmetric loss function is introduced by Varian (1975), called LINEX loss function, has been advocated by Zellner (1986), Basu and Ebrahimi (1991), Soliman (2002), and Singh et al. (2005). An alternative to the modified LINEX loss function is the general entropy loss function (GELF) proposed by Calabria and Pulcini in 1994. This function rises exponentially on one side of zero and becomes roughly linear on the opposite side.

In this paper, the ISB p-dim Rayleigh distribution with MLEs and Bayes estimator procedure for the one parameter under squared error, LINEX and general entropy loss functions, using type-II censored data are derived. We present the derivation of the maximum likelihood estimation of the unknown parameter in section 2. In Section 3, we develop the Bayes estimator of the unknown parameter. The approximate Bayes estimator based on Lindley's approximation, under square error, LINEX and general entropy loss functions are also considered in sections 4. Simulation and conclusion study is provided for numerical results in section 4.

2. Maximum Likelihood Estimator

Let $y_1,y_2,\ldots,y_m$ be the lifetimes of the *m* units subjected to life test. With type II censoring plan, one observes the primary *m* order statistics, $y_1\le y_2\le\ldots\le y_m$ from the sample $y_1,y_2,\ldots,y_n$. Based on the censored data $y_1,y_2,\ldots,y_m$ the likelihood function is

$$l(\underline{y}|\alpha)=\frac{n!}{(n-m)!}\prod_{i=1}^{m}f(y_i)\left[1-F(y_{(m)})\right]^{n-m}\qquad(2.1)$$

Using equations (1.1) and (1.2) the likelihood function (2.1) can be expressed as

$$l(\underline{y}|\alpha)=\frac{n!}{(n-m)!}\frac{2^m}{\alpha^{m(p+1)/2}\left(\Gamma\left(\frac{p+1}{2}\right)\right)^m}\prod_{i=1}^{m}\left(\frac{1}{y_i^{p+2}}\right)\exp\left(-\sum_{i=1}^{m}\left(\frac{1}{\alpha y_i^2}\right)\right)$$

$$\times\left\{1-\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\Gamma\left(\frac{p+1}{2},\frac{1}{\alpha\, y_m^2}\right)\right\}^{n-m}\qquad(2.2)$$

where, $\underline{y}=(y_1,y_2,\ldots\ldots,y_m)$. Natural logarithmic of the likelihood function is taken for equation (2.2) and after simplification; the corresponding log likelihood function is given by

$$\ln l=\ln(A)+m\ln(2)-mP\ln(\alpha)-m\ln(\Gamma P)+K_2-\frac{1}{\alpha}K_1+(n-m)\ln\left\{\Gamma P-\Gamma\left(P,\frac{1}{\alpha}K_3\right)\right\}\qquad(2.3)$$

where, $K_1=\sum_{i=1}^{m}\left(\frac{1}{y_i^2}\right)$, $K_2=\sum_{i=1}^{m}\ln\left(\frac{1}{y_i^{p+2}}\right)$, $K_3=\frac{1}{y_m^2}$, $P=\left(\frac{p+1}{2}\right)$

The MLE of *α* obtained by the setting the first partial derivatives of equation (2.3) equal to zero with respective to *α*, respectively, these simultaneous equations is,

$$\frac{\partial\ln l}{\partial\alpha}=-mP\frac{1}{\alpha}+\frac{1}{\alpha^2}K_1-(n-m)\Phi\frac{\Gamma\left(P,\frac{1}{\alpha}K_3\right)}{\left\{\Gamma P-\Gamma\left(P,\frac{1}{\alpha}K_3\right)\right\}}\qquad(2.4)$$

where, $\Phi\Gamma\left(P,\frac{1}{\alpha}K_3\right)=\dfrac{\partial\Gamma\left(P,\frac{1}{\alpha}K_3\right)}{\partial\alpha}$

where, $\hat{\alpha}$ represent of MLE. Newton Raphson iterative algorithm is used to compute the sample estimate $\hat{\alpha}$ as equation (2.4) cannot be solved analytically.

2.1 Asymptotic Confidence Interval

To obtain the asymptotic confidence interval (ACI), Fisher information matrix *I(α)* is used and given as:

$$I(\hat{\alpha}) = -E\left(\frac{\partial^2 \ln l}{\partial \alpha^2}\right)_{\alpha=\hat{\alpha}}$$

Now, the ACI of α is given by

$$\mathrm{var}(\hat{\alpha}) = \frac{1}{I(\hat{\alpha})}$$

Exact mathematical expression for the above expectation does not exist; hence, the concept of large sample theory is applied. $100\left(1-\frac{\mu}{2}\right)\%$ Confidence interval of the unknown parameter α is given by

$$[\hat{\alpha}_L, \hat{\alpha}_U] = \hat{\alpha}_{ML} \pm Z_{\mu/2}\sqrt{\mathrm{var}(\hat{\alpha}_{ML})}$$

where, $Z_{\mu/2}$ is the upper $(\mu/2)^{th}$ percentile of the standard normal distribution.

3. Bayesian Estimation

This section deals with Bayes estimate for unknown parameter α under squared error, LINEX and general entropy loss functions.

Prior distribution of parameter α is taken as natural conjugate prior is taken as non-informative uniform prior by which is of the form

$$g(\alpha) = \frac{c^\delta}{\Gamma(\delta)}\alpha^{\delta-1}\exp(-c\alpha);\ \alpha, \delta, c > 0 \tag{3.1}$$

Now the joint posterior density function $\prod\left(\underline{y}\middle|\alpha\right)$ as from equation (2.2) and (3.1) we get,

$$\prod\left(\underline{y}\middle|\alpha\right) = \frac{L(\underline{y}|\alpha)g(\alpha)}{\int_\alpha L(\underline{y}|\alpha)g(\alpha)\,d\alpha}$$

$$\prod\left(\underline{y}\middle|\alpha\right) = \frac{2^m c^\delta K^{-1}\exp(-c\alpha)}{\alpha^{(m(p+1)+(\delta-1))/2}\Gamma(\delta)\left(\Gamma\left(\frac{p+1}{2}\right)\right)^m}\prod_{i=1}^{m}\left(\frac{1}{y_i^{p+2}}\right)\exp\left(-\sum_{i=1}^{m}\left(\frac{1}{\alpha y_i^2}\right)\right)\left\{1-\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\Gamma\left(\frac{p+1}{2}, \frac{1}{\alpha\, y_i^2}\right)\right\}^{(n-m)} \tag{3.2}$$

Where,

$$K = \int_\alpha \frac{2^m c^\delta \exp(-c\alpha)}{\alpha^{(m(p+1)+(\delta-1))/2}\Gamma(\delta)\left(\Gamma\left(\frac{p+1}{2}\right)\right)^m}\prod_{i=1}^{m}\left(\frac{1}{y_{i:m:n}^{p+2}}\right)\exp\left(-\sum_{i=1}^{m}\left(\frac{1}{\alpha y_{i:m:n}^2}\right)\right)\times\prod_{i=1}^{m}\left\{1-\frac{1}{\Gamma\left(\frac{p+1}{2}\right)}\Gamma\left(\frac{p+1}{2}, \frac{1}{\alpha\, y_{i:m:n}^2}\right)\right\}^{R_i} d\alpha$$

In the next section, the well-known method, Lindley's Approximation is used to obtain Bayes estimates of individual parametric function in closed form.

3.1 Lindley's Approximation Method

In this paper, we consider a problem of ISB p-dim Rayleigh distribution using Lindley's approximation method for bayes estimator. The approach developed by Lindley (1980), provides a simplified form of bayes estimator which is easy to use in practice. Lindley's Approximation method I(y) can be approximated as

$$I(\underline{y}) = \frac{\int_0^\infty p(\alpha)e^{l(\alpha/\underline{y})+\eta(\alpha)}d\alpha}{\int_0^\infty e^{l(\alpha/\underline{y})+\eta(\alpha)}d\alpha} \tag{3.3}$$

where, $p(\alpha)$ = function of α,

$l(\alpha|\underline{y})$ = the log likelihood function,

$\eta(\alpha,\beta)$ = log of prior distribution of α.

$$I(\underline{y}) = p(\hat{\alpha}) + \frac{1}{2}\left[\left(\hat{p}_{\alpha\alpha} + 2\hat{p}_\alpha\hat{\eta}_\alpha\right)\hat{\sigma}_{\alpha\alpha}\right] + \frac{1}{2}\left[\hat{p}_\alpha\hat{\sigma}_{\alpha\alpha}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \tag{3.4}$$

Where $\hat{\alpha}$ is MLE of α, $p_\alpha = \frac{\partial p(\alpha)}{\partial\alpha}, p_{\alpha\alpha} = \frac{\partial^2 p(\alpha)}{\partial\alpha^2}, \hat{l}_\alpha = \frac{\partial l}{\partial\alpha}, \hat{l}_{\alpha\alpha} = \frac{\partial^2 l}{\partial\alpha^2}, \hat{l}_{\alpha\alpha\alpha} = \frac{\partial^3 l}{\partial\alpha^3}$

$$\hat{\eta}_\alpha = \left.\frac{\partial \log g(\alpha)}{\partial\alpha}\right|_{\alpha=\hat{\alpha},} = \frac{(\delta-1)}{\hat{\alpha}} - c,\ \sigma_{\alpha\alpha} = -\frac{1}{l_{\alpha\alpha}}$$

Now, the Bayes estimator under the SELF of the parameter α from equation (3.4) is computed in the following forms:

(i) Bayes estimate of α under SELF

If p (α) =α, then $p_\alpha = 1, p_{\alpha\alpha} = 0$, the Bayes estimate of α under SELF is given by

$$\tilde{\alpha}_{BS} = E\left(\alpha/\underline{y}\right) = \hat{\alpha}_{ML} + 0.5\left[\left(\hat{p}_{\alpha\alpha} + 2\hat{p}_\alpha\hat{\eta}_\alpha\right)\hat{\sigma}_{\alpha\alpha}\right] + 0.5\left[\hat{p}_\alpha\hat{\sigma}_{\alpha\alpha}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \tag{3.5}$$

The Bayes estimator under the LINEX loss function of the parameter α from equation (3.4) is compute. We obtain the following form:

(ii) Bayes estimate of Parameter α under LINEX

If $p(\alpha) = e^{-k\alpha}$,then $p_\alpha = -ke^{-k\alpha}, p_{\alpha\alpha} = k^2e^{-k\alpha}$.

Bayesian estimate of α under LINEX loss function is given by

$$\tilde{\alpha}_{BLL} = -\frac{1}{k}\ln E_\alpha\left(e^{-k\alpha}\right)$$
$$E\left(e^{-k\alpha}/\underline{y}\right)= e^{-k\hat{\alpha}_{ML}} + 0.5\left[\left(\hat{p}_{\alpha\alpha}+2\hat{p}_\alpha\hat{\eta}_\alpha\right)\hat{\sigma}_{\alpha\alpha}\right] + 0.5\left[\hat{p}_\alpha\hat{\sigma}_{\alpha\alpha}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \tag{3.6}$$

Next, the Bayes estimator under the general entropy loss function of the parameters α from equations (3.4) is compute. The following form is obtained:

(iii) Bayes estimate of α under GELF

If $p(\alpha)=\alpha^{-q}$,then $p_\alpha = -q\alpha^{-(q+1)}$, $p_{\alpha\alpha} = q(q+1)\alpha^{-(q+2)}$,the Bayes estimate of α under GELF loss function is given by

$$\tilde{\alpha}_{BG} = \left[E_\alpha\left(\alpha^{-q}/\underline{y}\right)\right]^{-1/q},$$
$$E_\alpha\left(\alpha^{-q}/\underline{y}\right)= \hat{\alpha}^{-q} + 0.5\left[\left(\hat{p}_{\alpha\alpha}+2\hat{p}_\alpha\hat{\eta}_\alpha\right)\hat{\sigma}_{\alpha\alpha}\right] + 0.5\left[\hat{p}_\alpha\hat{\sigma}_{\alpha\alpha}\left(\hat{l}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \tag{3.7}$$

Simulation Study and conclusion

Mathematical expression for maximum likelihood and Bayes estimators of unknown parameter for ISB p-dim Rayleigh distribution under type-II censoring is obtained in this paper. MLE and each of the proposed Bayes estimate is empirically computed based on Lindley approximation with three loss functions (square error, LINEX and general entropy). In our simulation study we have generated a sample of size n=20, 40, 60 to observe the effect of small, median and large samples on the estimators. The results are replicated 5000 times and the average of the results has been presented in the tables.

Table. 1: Average values of ML and Bayes estimate of the parameter α with respective MSEs (in Bracket) under for different values of (*n*, *m*).

n	*m*	$\hat{\alpha}_{ML}$	$\tilde{\alpha}_{BSL}$	$\tilde{\alpha}_{BL_1L}$	$\tilde{\alpha}_{BL_2L}$	$\tilde{\alpha}_{BG_1L}$	$\tilde{\alpha}_{BG_2L}$
20	10	1.5264 (0.0613)	1.5082 (0.0343)	1.4485 (0.0313)	1.5563 (0.0372)	1.4927 (0.0335)	1.5736 (0.0437)
	20	1.5836 (0.0568)	1.5537 (0.0315)	1.5234 (0.0268)	1.5864 (0.0365)	1.5864 (0.0287)	1.6114 (0.0395)
40	10	1.4667 (0.0624)	1.4475 (0.0436)	1.4126 (0.0374)	1.4586 (0.0469)	1.4398 (0.0385)	1.4624 (0.0527)
	20	1.5625 (0.0603)	1.5374 (0.0381)	1.4196 (0.0344)	1.5063 (0.0423)	1.4636 (0.0396)	1.5496 (0.0469)
	30	1.6281 (0.0569)	1.5864 (0.0326)	1.4903 (0.0265)	1.5863 (0.0358)	1.5299 (0.0262)	1.6327 (0.0416)
	40	1.6673 (0.0523)	1.6083 (0.0278)	1.5779 (0.0229)	1.6492 (0.0335)	1.5906 (0.0268)	1.6738 (0.0347)
60	10	1.4637 (0.0615)	1.4262 (0.0371)	1.3736 (0.0319)	1.4898 (0.0386)	1.4137 (0.0355)	1.4974 (0.0435)
	20	1.4743 (0.0603)	1.4581 (0.0346)	1.3875 (0.0275)	1.4672 (0.0384)	1.4265 (0.0332)	1.4765 (0.0415)
	30	1.5378 (0.0565)	1.4783 (0.0327)	1.4266 (0.0258)	1.5135 (0.0359)	1.4395 (0.0285)	1.5287 (0.0387)
	40	1.5852 (0.0526)	1.5562 (0.0279)	1.5285 (0.0238)	1.5735 (0.0327)	1.5398 (0.0254)	1.5812 (0.0368)
	50	1.6276 (0.0487)	1.5824 (0.0245)	1.5426 (0.0216)	1.5868 (0.0305)	1.5545 (0.0227)	1.6134 (0.0321)
	60	1.6863 (0.0431)	1.6175 (0.0215)	1.5658 (0.0185)	1.6357 (0.0253)	0.5814 (0.0209)	1.6735 (0.0316)

The Bayes estimate based on Type-II censored data relative to square error $\tilde{\alpha}_{BSL}$, LINEX $\tilde{\alpha}_{BLL}$ and general entropy $\tilde{\alpha}_{BGL}$ loss functions are found to be more efficient in terms of having lower MLE and hence, are regarded as being superior to MLE, for all permutations of *n* and *m*.

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