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Cubic AT-Subalgebras and AT-Ideals on AT-Algebra

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Abstract
In this paper, the notions of cubic AT-ideals and cubic AT-subalgebras in AT-algebras are introduced and several properties are investigated. The image and inverse image of them in AT-algebras are defined and studied.

Keywords: AT-algebras, cubic AT-ideals, cubic AT-sub algebras, homomorphism of cubic AT-ideals.

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1. Introduction
K. Is´eki and S. Tanaka ([5]) studied ideals and congruences of BCK-algebras. S. M. Mostafa and et al. ([1],[9]) were introduced a new algebraic structure which is called KUS-algebras and investigated some related properties. The concept of a fuzzy set, was introduced by L.A. Zadeh [6]. O.G. Xi [8] applied the concept of fuzzy set to BCK-algebras and gave some of its properties. Y. B. Jun and et al. [10] Were introduced the notion of cubic ideals in BCK-algebras, and they discussed some related properties of it. In ([3]), Areej Tawfeeq Hameed and et al. introduced the notion of cubic KUS-ideals of KUS-algebra and they were studied the homomorphic image and inverse image of cubic KUS-ideals. In this paper, we introduce the notion of cubic AT-ideals of AT-algebra and we study the homomorphic image and inverse image of cubic AT-ideals of AT-algebra.

2. Preliminaries
In this section, we give some basic definitions and preliminaries proprieties of AT-ideals and fuzzy AT-ideals in AT-algebra such that we include some elementary aspects that are necessary for this paper.

Definition 2.1[2]. An **AT-algebra** is a nonempty set X with a constant (0) and a binary operation (*) satisfying the following axioms: for all x, y, z$\in$X,
(i) $(x*y)*((y*z)*(x*z))=0$,
(ii) $0*x=x$,
(iii) $x*0=0$.
In X, we can define a binary relation ($\leq$) by: $x \leq y$ if and only if, $y * x = 0$.

Example 2.2 [2]. Let X = {0, 1, 2, 3, 4} in which (*) is defined by the following table:

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

It is easy to show that (X ;*, 0) is an AT-algebra.

Proposition 2.3 [2]. In any AT-algebra (X ;*, 0), the following properties holds: for all x, y, z ∈X;
a) z * z = 0,
b) x = 0 *(0*x),
c) z*(x *z) = 0,
d) y * ((y* z) * z) = 0,
e) x * y = 0 implies that x * 0 = y * 0,
f) 0*x=0*y implies that x=y.

Proposition 2.4[2]. In any AT-algebra (X ;*, 0), the following properties holds: for all x, y, z ∈X;
a) x ≤ y implies that y ∗ z ≤ x ∗ z,
b) x ≤ y implies that z ∗ x ≤ z ∗ y,
c) x ∗ y ≤ z imply z ∗ y ≤ x
d) (y ∗z) ∗(x ∗z) ≤ x ∗y,
e) z ∗ x ≤ z ∗ y implies that x ≤ y(left cancellation law).

Definition 2.5[2]. A nonempty subset S of an AT-algebra (X ;∗, 0)is called an AT-subalgebra of AT-algebra X if for all x, y ∈ S, then x ∗ y ∈ S.

Definition 2.6[2]. A nonempty subset I of an AT-algebra (X ;∗, 0)is called an AT-ideal of AT-algebra X if it satisfies the following conditions: for all x, y, z ∈ X.
AT_1) 0 ∈ I ;
AT_2) x ∗ (y ∗ z) ∈ I and y ∈ I imply x ∗ z∈ I.

Definition 2.7[6]. Let X be a nonempty set, a fuzzy subsetμ in X is a function μ : X → [0,1].

Definition 2.8[7]. Let X be a set and μ be a fuzzy subset of X, for t ∈[0,1], the set
μ_t={x ∈X| μ(x) ≥ t}is called a level subset of μ.

Definition 2.9[2]. Let (X ;*, 0) be an AT-algebra. A fuzzy set μ in X is called a fuzzy AT-subalgebra of X if for all x, y ∈ X, then μ (x ∗ y) ≥ min { μ (x), μ (y)}.

Definition 2.10[2]. Let (X ;*, 0) be an AT-algebra. A fuzzy set μ in X is called **a fuzzy AT-ideal of X** if it satisfies the following conditions: for all x, y and z ∈ X,
(AT_1)μ (0) ≥μ (x).
(AT_2)μ (x ∗ z) ≥min { μ (x*(y ∗ z)),μ (y)}.

Definition 2.11[4]. Let (X;∗, 0) and (Y;∗`,0`) benonempty sets. The mapping f : (X;∗,0) → (Y;∗`,0`) is called a homomorphism if it satisfies f (x ∗ y) = f (x)∗` f (y), for all, y ∈X. The set {x∈X| f (x) = 0'} is called the kernel of f and is denoted by ker f .

Definition 2.12[4]. Let f : (X;∗,0) →(Y;∗',0') be a mapping from the set X to a set Y. If μis a fuzzy subset of X, then the fuzzy subset *f*(μ) in Y defined by:

$$f(\mu)(y)=\begin{cases} \sup\{\mu(x): x\in f^{-1}(y)\} & \text{if } f^{-1}(y)=\{x\in X, f(x)=y\}\neq\phi \\ 0 \quad \text{otherwise} \end{cases}$$

is said to be the image of μ under *f*.

Similarly if β is a fuzzy subset of Y, then the fuzzy subset μ = (β °*f*) in X, (i.e the fuzzy subset defined by μ (x) = β(*f*(x)) for all x ∈X) is called the pre-image of βunder *f*.

Theorem 2.13[2]. Let *f*: (X;∗,0) → (Y;∗`,0`) be a homomorphism of AT-algebras, then :
(F_1) f (0) = 0'.
(F_2) If S is an AT-subalgebra of X, then f (S) is an AT-subalgebra in Y, where *f* is onto.
(F_3) If B is an AT-subalgebra in Y, then f^{-1} (B) is an AT-subalgebra in X.
(F_4) If I is an AT-ideal of X, then f (I) is an AT-ideal in Y, where *f* is onto.
(F_5) If J is an AT- ideal in Y, then f^{-1} (J) is an AT-ideal in X.
(F_6) f is injective if and only if, ker f = {0}.

Now, we will recall the concept of interval-valued fuzzy subsets.

Remark 2.14[3, 10]. An interval number is $\tilde{a} = [a^-, a^+]$, where 0 ≤ a^-≤ a^+≤ 1. Let I be a closed unit interval, (i.e., I = [0, 1]). Let D[0, 1] denote the family of all closed subintervals ofI = [0, 1], that is, D[0, 1] = { $\tilde{a} = [a^-, a^+]$|a^-≤ a^+, for a^-,a^+∈ I}.
Now, we define what is known as refined minimum (briefly, rmin) of two element in D [0,1].

Definition 2.15[3,10]. We also define the symbols (≽), (≼), (=),"rmin " and "rmax " in case of two elements in D[0, 1]. Consider two interval numbers (elements numbers)
$\tilde{a} = [a^-, a^+]$, $\tilde{b} = [b^-, b^+]$in D[0, 1] : Then
(1) $\tilde{a}$≽$\tilde{b}$if and only if, a^-≥b^- and a^+≥b^+,
(2) $\tilde{a}$≼$\tilde{b}$if and only if, a^-≤b^- and a^+≤b^+,
(3) $\tilde{a}$=$\tilde{b}$if and only if, a^-=b^- and a^+=b^+,
(4) rmin {$\tilde{a}$, $\tilde{b}$}= [min {a^-,b^-}, min {a^+,b^+}],
(5) rmax {$\tilde{a}$, $\tilde{b}$}= [max {a^-,b^-}, max {a^+,b^+}],

Remark 2. 16[3,10]. It is obvious that (D[0, 1], ≼, ∨, ∧) is a complete lattice with$\tilde{0}$ = [0,0] as its least element and $\tilde{1}$ = [1, 1] as its greatest element. Let $\tilde{a}_i$ ∈D[0, 1] where i∈Λ. We define$\mathrm{rinf}_{i\in\Lambda}\tilde{a}$= [r $\inf_{i\in\Lambda}a^-$, $\mathrm{rinf}_{i\in\Lambda}a^+$], $\mathrm{rsup}_{i\in\Lambda}\tilde{a}$= [$\mathrm{rsup}_{i\in\Lambda}a^-$, $\mathrm{rsup}_{i\in\Lambda}a^+$].

Definition 2.17[3,10]. An interval-valued fuzzy subset $\tilde{\mu}_A$on X is defined as
$\tilde{\mu}_A$ ={< x, [μ_A^-(x),μ_A^+(x)]>| x∈ X}. Where μ_A^-(x) ≤μ_A^+(x), for all x∈ X. Then the ordinary fuzzy subsetsμ_A^-: X → [0, 1] and μ_A^+: X → [0, 1] are called a lower fuzzy subset and an upper fuzzy subset of $\tilde{\mu}_A$ respectively. Let $\tilde{\mu}_A$ (x) = [μ_A^-(x),μ_A^+(x)],
$\tilde{\mu}_A$:X → D[0, 1], then A = {< x, $\tilde{\mu}_A$ (x) >| x∈ X}.

Definition 2.18([10]). Let (X ;*, 0) be a nonempty set. A cubic set Ω in a structure Ω = {< x, $\tilde{\mu}_\Omega$ (x),λ_Ω (x)>| x∈ X}, which is briefly denoted by Ω =<$\tilde{\mu}_\Omega$, λ_Ω>, where $\tilde{\mu}_\Omega$:X → D[0, 1],$\tilde{\mu}_\Omega$ is an interval-valued fuzzy subset of X and λ_Ω:X → [0, 1],λ_Ω is a fuzzy subset of X.

Definition 2.19([10]). For any$\Omega_i = \{\langle x, \tilde{\mu}_{\Omega i}(x), v_{\Omega i}(x)|x \in X\rangle\}$where i∈Λ,p-union and p-intresection is denoted by$\bigcup_{i\in\Lambda\ p}\Omega_i$and $\bigcap_{i\in\Lambda\ p}\Omega_i$and is defined respectively by:-

$$\bigcup_{i\in\Lambda\ p}\Omega_i = \left\{\langle x, \left(\bigcup_{i\in\Lambda}\tilde{\mu}_{\Omega i}\right)(x), \left(\bigvee_{i\in\Lambda} v_{\Omega i}\right)(x)|x \in X\rangle\right\}, \bigcap_{i\in\Lambda\ p}\Omega_i = \left\{\langle x, \left(\bigcap_{i\in\Lambda}\tilde{\mu}_{\Omega i}\right)(x), \left(\bigwedge_{i\in\Lambda} v_{\Omega i}\right)(x)|x \in X\rangle\right\}.$$

Definition 2.19([10]). For a family $\Omega_i = \{\langle x, \tilde{\mu}_{\Omega i}(x)\rangle|x \in X\}$on fuzzy sets in X}where i∈Λ and Λ is index set, we define the join (∨)and meet (∧) operations as follows:

$$\bigvee_{i\in\Lambda}\Omega_i = \left(\bigvee_{i\in\Lambda}\tilde{\mu}_{\Omega i}\right)(x) = \sup\{\tilde{\mu}_{\Omega i}(x)|i \in \Lambda\}, \bigwedge_{i\in\Lambda}\Omega_i = \left(\bigwedge_{i\in\Lambda}\tilde{\mu}_{\Omega i}\right)(x) = \inf\{\tilde{\mu}_{\Omega i}(x)|i \in \Lambda\},$$

3. Cubic AT-subalgebras of AT-algebras

In this section, we will introduce a new notion called cubic AT-subalgebra of AT-algebras and study several properties of it.

Definition 3.1. Let (X ;*, 0)be an AT-algebra. A cubic set Ω =<$\tilde{\mu}_\Omega$ (x),λ_Ω(x) > of X is called cubic AT-subalgebra of X if, for all x, y, z ∈X:
$\tilde{\mu}_\Omega$ (x ∗ z) ≽ rmin{$\tilde{\mu}_\Omega$ (x), $\tilde{\mu}_\Omega$ (y)}, and λ_Ω (x*y) ≤ max{λ_Ω (x), λ_Ω (y)}.

Example 3.2. Let X = {0,1,2,3} in which the operation as in example (∗) be define by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then (X;∗,0) is an AT-algebra. Define a cubic set Ω =<$\tilde{\mu}_\Omega$, λ_Ω> of X as follows:
fuzzy subset μ: X⟶ [0,1] by: $\tilde{\mu}_\Omega$ (x) =

$$\begin{cases} [0.3,0.9] & if\ x = \{0,1\} \\ [0.1,0.6] & otherwise \end{cases}$$

and λ_Ω= $\begin{cases} 0.1 & if\ x = \{0,1\} \\ 0.6 & otherwise \end{cases}$ .The cubic set Ω =<$\tilde{\mu}_\Omega$, λ_Ω> is a cubic AT-subalgebra of X.

Proposition 3.3. Let Ω =<$\tilde{\mu}_\Omega$, λ_Ω> be a cubic AT-subalgebra of AT-algebra (X ;*, 0), then$\tilde{\mu}_\Omega$ (0)≽$\tilde{\mu}_\Omega$ (x) and λ_Ω (0)≤λ_Ω (x), for all x∈X.

Proof. For all x∈X, we have
$\tilde{\mu}_\Omega$ (0) = $\tilde{\mu}_\Omega$ (x *x) ≽ rmin{$\tilde{\mu}_\Omega$ (0*(0*x)), $\tilde{\mu}_\Omega$ (x)}
= rmin{Ω (0*(0*x)), $\tilde{\mu}_\Omega$ (x)}= rmin {[μ_A^- (x),μ_A^+ (x)],[μ_A^- (x),μ_A^+ (x)]}
= rmin {[μ_A^- (x),μ_A^+ (x)]}=$\tilde{\mu}_\Omega$ (x).
Similarly, we can show that λ_Ω (0) ≤ max {[λ_Ω (x),λ_Ω (x)]} =λ_Ω (x). ⌂

Proposition 3.4. If a cubic set Ω=($\tilde{\mu}_\Omega$,v_Ω) of X is a cubic AT-subalgebra, then
Ω(x ∗ y)= Ω(x ∗ (0 ∗ (0 ∗ y))), for all x, y ∈ X.

Proof.
Let X be an AT-algebra and x,y ∈X, then we know that y=0∗(0∗y). Hence,
$\tilde{\mu}_\Omega$(x ∗ y)=$\tilde{\mu}_\Omega$(x ∗ (0 ∗ (0 ∗ y))) and v_Ω(x ∗ y)=v_Ω(x ∗ (0 ∗ (0 ∗ y))). Therefore
Ω(x ∗ y)= Ω(x ∗ (0 ∗ (0 ∗ y))). ⌂

Theorem 3.5. Let (X ;*, 0) be an AT-algebra and A cubic set Ω =<$\tilde{\mu}_\Omega$, λ_Ω> of X. A cubic set Ω of X is a cubic AT-subalgebra of X if and only if, μ_A^-,$\mu_A^+$and $\lambda_\Omega$are cubic AT-subalgebras of X.

Proof. If $\mu_A^-$and $\mu_A^+$are cubic AT-subalgebras of X. For any x, y ∈X. Observe
$\tilde{\mu}_\Omega$ (x ∗ y) = [μ_A^- (x*y),μ_A^+ (x ∗ y)] ≽ [min {μ_A^- (x), μ_A^- (y)}, min {μ_A^+ (x), μ_A^+ (y)}]
= rmin {[μ_A^- (x),μ_A^+ (x], [μ_A^- (y), μ_A^+ (y)]} = rmin {$\tilde{\mu}_\Omega$ (x),$\tilde{\mu}_\Omega$ (y)].
Similarly, we can show that λ_Ω (x ∗ y) ≤ max {[λ_Ω (x),λ_Ω (y)]}.
From what was mentioned above we can conclude that Ω is a cubic AT-subalgebra of X.

Conversely, suppose that Ω is a cubic AT-subalgebra of X. For all x, y ∈X, we have
[μ_A^- (x ∗ y), μ_A^+ (x ∗ y)] =$\tilde{\mu}_\Omega$ (x ∗ z) ≽ rmin{$\tilde{\mu}_\Omega$ (x),$\tilde{\mu}_\Omega$ (y)}
= rmin{[μ_A^- (x),μ_A^+ (x)], [μ_A^- (y), μ_A^+ (y)]}= [min{μ_A^- (x),μ_A^- (y) }, min {μ_A^+ (x),μ_A^+ (y)}].
Therefore, μ_A^- (x ∗ y) ≥ min{μ_A^- (x),μ_A^- (y)} andμ_A^+ (x ∗ y) ≥ min{μ_A^+ (x),μ_A^+ (y)}.
Similarly, we can show that λ_Ω (x ∗ y)≤max{λ_Ω (x),λ_Ω (y)}
Hence, we get that μ_A^-, $\mu_A^+$and $\lambda_\Omega$are cubic AT-subalgebras of X. ⌂

Theorem 3.6. The R-intersection of any set of cubic AT-subalgebra of X is also cubic AT-subalgebra of X.

Proof. Let Ω_i ={⟨x, $\tilde{\mu}_{\Omega i}$, (x), $v_{\Omega i}$(x)⟩|x ∈ X}wheri ∈ Λ, be a set of cubic AT-subalgebra of Xand x, y ∈ X, then
$(\cap \tilde{\mu}_{\Omega i})(x * y)$ = rinf $\tilde{\mu}_{\Omega i}(x * y)$ ≥rinf {rmin{$\mu_{\Omega i}(x), \mu_{\Omega i}(y)$}}
=rmin{rin($\mu_{\Omega i}$(x)), rinf ($\mu_{\Omega i}$(y))} =
rmin{$(\cap \tilde{\mu}_{\Omega i})(x), (\cap \tilde{\mu}_{\Omega i})(y)$}and
$\left(\bigvee v_{\Omega i}\right)(x * y) = \sup v_{\Omega i}\ (x * y) \le \sup\{\max\{v_{\Omega i}(x), v_{\Omega i}(y)\}\}$
=max{sup($v_{\Omega i}$(x)),sup($v_{\Omega i}$(y))}= max{$(\vee v_{\Omega i})(x), (\vee v_{\Omega i})(y)$}.⌂

Which shows that R-intresection works as a cubicAT-subalgebra of X.

Theorem 3.7. The R-intresection of any set of cubic AT-subalgebra of Xis also cubic subalgebra of X.

Proof. Let $\Omega_i = \{\langle x, \tilde{\mu}_{\Omega i}(x), v_{\Omega i}(x)\rangle | x \in X\}$where $i \in \Lambda$, be a set of cubic AT-subalgebra of Xand $x, y \in X$, then

$(\cap \tilde{\mu}_{\Omega i})(x * y) = \text{rinf}\, \tilde{\mu}_{\Omega i}(x * y) \geq \text{rinf}\, \{\text{rmin}\{\mu_{\Omega i}(x), \mu_{\Omega i}(y)\}\}$
$= \text{rmin}\{\text{rinf}(\mu_{\Omega i}(x)), \text{rinf}\, (\mu_{\Omega i}(y))\} =$
$\text{rmin}\{(\cap \tilde{\mu}_{\Omega i})(x), (\cap \tilde{\mu}_{\Omega i})(y)\}$and
$\left(\bigvee v_{\Omega i}\right)(x * y) = \sup v_{\Omega i}\,(x * y) \leq \sup\{\max\{v_{\Omega i}(x), v_{\Omega i}(y)\}\}$
$= \max\{\sup(v_{\Omega i}(x)), \sup(v_{\Omega i}(y))\} = \max\{(\vee v_{\Omega i})(x), (\vee v_{\Omega i})(y)\}$.⌂

Remark 3.8. The R–union, p-intresection and p-union of any sets of cubic AT-subalgebra need not be a cubic AT-subalgebra, for example:

Example 3.9.
Let X={0,a,b,c,d,e} be AT-subalgebra with the following cayley table.

*	0	a	b	c	d	e
0	0	b	a	c	d	e
a	a	0	b	e	c	d
b	b	a	0	d	e	c
c	c	d	e	0	a	b
d	d	e	c	b	o	a
e	e	c	d	a	b	0

We defined two cubic set $\Omega_1 = (\tilde{\mu}_{\Omega 1}, v_{\Omega 1})$ and $\Omega_2 = (\tilde{\mu}_{\Omega 2}, v_{\Omega 2})$ of X by :-

$\tilde{\mu}_{\Omega 1}(x) = \begin{cases}[0.6,0.7], \text{if} x \in \{0, c\}, \\ [0.1,0.2], \text{otherwise},\end{cases}$ $v_{\Omega 1}(x) = \begin{cases}0.2, \text{if} x \in \{0, c\}, \\ 0.6, \text{otherwise},\end{cases}$

$\tilde{\mu}_{\Omega 2}(x) = \begin{cases}[0.8,0.9], \text{if} x \in \{0, d\}, \\ [0.3,0.4], \text{otherwise},\end{cases}$ and $v_{\Omega 2}(x) = \begin{cases}0.1, \text{if} x \in \{0, c\}, \\ 0.4, \text{otherwise}.\end{cases}$

Then Ω_1 and Ω_2 are cubic AT-subalgebra of X but R – union, p-intresection and p-union of $\Omega_1$and Ω_2 are not cubic AT-subalgebras of X.

Since$(\cup \tilde{\mu}_{\Omega i})(c * d) = [0.3,0.4] \ngeq [0.6,0.7] = \text{rmin}\{(\cup \tilde{\mu}_{\Omega i})(c), (\cup \mu_{\Omega i})(d)\}$and $(\wedge \tilde{\mu}_{\Omega i})(c * d) = 0.4 \nleq 0.2 = \max\{(\wedge \tilde{\mu}_{\Omega i})(c), (\wedge \tilde{\mu}_{\Omega i})(d)\}$.

Theorem 3.10. Let $\Omega i = (\tilde{\mu}_{\Omega i}, v_{\Omega i})$ be a cubic AT-subalgebra of X, where $i \in \Lambda$
$\inf\{\max\{v_{\Omega i}(x), v_{\Omega i}(x)\}\} = \max\{\inf v_{\Omega i}(x), \inf v_{\Omega i}(x)\}$,for all $x \in X$, then thep-intresection of Ω_i is also a cubic one of X

Proof. Let $\Omega_i = \{\langle x, \tilde{\mu}_{\Omega i}, (x), v_{\Omega i}(x)\rangle | x \in X\}$wheri $\in \Lambda$,be a set of cubic AT-subalgebra of X such that $\inf\{\max\{v_{\Omega i}(x), v_{\Omega i}(x)\}\} = \max\{\inf v_{\Omega i}(x), \inf v_{\Omega i}(x)\}$for all $x \in X$, then for $x, y \in X$,

$$(\cap \tilde{\mu}_{\Omega i})(x * y) = \text{rinf}\, \tilde{\mu}_{\Omega i}(x * y) \geq \text{rinf}\{\text{rmin}\{\mu_{\Omega i}(x), \mu_{\Omega i}(y)\}\}$$
$$= \text{rmin}\{\text{rinf}\, \mu_{\Omega i}(x), \text{rinf}\, \mu_{\Omega i}(y)\} = \text{rmin}\{(\cap \tilde{\mu}_{\Omega i})(x), (\cap \tilde{\mu}_{\Omega i})(y)\}$$
$$\text{and } (\wedge v_{\Omega i})(x * y) = \inf v_{\Omega i}\,(x * y) \leq \inf\{\max\{v_{\Omega i}(x), v_{\Omega i}(y)\}\}$$
$$= \max\{\inf v_{\Omega i}(x), \inf v_{\Omega i}(y)\} = \max\{(\wedge v_{\Omega i})(x), (\wedge v_{\Omega i})(y)\}.$$

Hence, p-intresection of Ω_i is a cubic AT-subalgebra of X. ⌂

Theorem 3.11. Let $\Omega i = (\tilde{\mu}_{\Omega i}, v_{\Omega i})$ be a cubic subalgebra of X where $i \in \Lambda$, for all $x \in X$
$\{\text{rmin}\{v_{\Omega i}(x), v_{\Omega i}(x)\}\} = \text{rmin}\{\text{rsup}\, v_{\Omega i}(x), \text{rsup} v_{\Omega i}(x)\}$, then thep-union of Ω_i is also a cubic one of X.

Proof. Let $\Omega_i = \{\langle x, \tilde{\mu}_{\Omega i}(x), v_{\Omega i}(x)\rangle | x \in X\}$, where $i \in \Lambda$,be a sets of cubic AT-subalgebras of X such that for all $x, y \in X$,

$$\text{rsup}\{\, \text{rmin}\{v_{\Omega i}(x), v_{\Omega i}(x)\}\} = \text{rmin}\{\text{rsup}\, v_{\Omega i}(x), \text{rsup}\, v_{\Omega i}(x)\}, \text{ then}$$
$$(\cup \tilde{\mu}_{\Omega i})(x * y) = \text{rsup} \tilde{\mu}_{\Omega i}(x * y) \geq \text{rsup}\{\text{rmin}\{\tilde{\mu}_{\Omega i}(x), \tilde{\mu}_{\Omega i}(y)\}\}$$
$$= \text{rmin}\{\text{rsup} \tilde{\mu}_{\Omega i}(x), \text{rsup} \tilde{\mu}_{\Omega i}(y)\} = \text{rmin}\,\{(\cup \tilde{\mu}_{\Omega i})(x), (\cup \tilde{\mu}_{\Omega i})(y)\}.$$
$$\left(\bigvee v_{\Omega i}\right)(x * y) = \sup v_{\Omega i}\,(x * y) \leq \sup\{\max\{v_{\Omega i}(x), v_{\Omega i}(y)\}\}$$
$$= \max\{\sup v_{\Omega i}(x), \sup v_{\Omega i}(y)\} = \max\{(\vee v_{\Omega i})(x), (\vee v_{\Omega i})(y)\},$$

Hence, p-union of Ω_i is a cubic AT-subalgebra of X. ⌂

Theorem 3.12. Let (X ;*, 0) be an AT-algebra. A cubic subset $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ of X, then Ω is a cubic AT-subalgebra of X if and only if, for all $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, the set $\tilde{U}(\Omega; \tilde{t}, s)$ is an AT-subalgebra of X, where $\tilde{U}(\Omega; \tilde{t}, s) = \{\, x \in X | \tilde{\mu}_\Omega(x) \succcurlyeq \tilde{t}, \lambda_\Omega(x) \leq s\}$.

Proof. Assume that $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$is a cubic AT-subalgebra of X and let $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, be such that $\tilde{U}(\Omega; \tilde{t}, s) \neq \emptyset$, and let $x, y \in X$ such that $x,y \in \tilde{U}(\Omega; \tilde{t}, s)$, then $\tilde{\mu}_\Omega(x) \succcurlyeq \tilde{t}$, $\tilde{\mu}_\Omega(y) \succcurlyeq \tilde{t}$ and $\lambda_\Omega(x), \leq s$, $\lambda_\Omega(y) \leq s$. By (A_2), we get
$\tilde{\mu}_\Omega(x * y) \succcurlyeq \min\{\, \tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$ and$\lambda_\Omega(x * y) \leq \max\{\lambda_\Omega(x), \lambda_\Omega(y)\} \leq s$.

Hence the set $\tilde{U}(\Omega; \tilde{t}, s)$ is an AT-subalgebra of X.

Conversely, suppose that $\tilde{U}(\Omega; \tilde{t}, s)$ is an AT-subalgebra of X and let $x, y \in X$ be such that$\tilde{\mu}_\Omega(x * y) \prec \text{rmin}\,\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$, and $\lambda_\Omega(x * y) > \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

Consider $\tilde{\beta} = 1/2\,\{\,\tilde{\mu}_\Omega(x * y) + \text{rmin}\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}\,\}$ and $\beta = 1/2\,\{\,\lambda_\Omega(x * y) + \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}\}$.

We have $\tilde{\beta} \in D[0, 1]$ and $\beta \in [0, 1]$, and $\tilde{\mu}_\Omega(x * y) \prec \tilde{\beta} \prec \text{rmin}\,\{\tilde{\mu}_\Omega(x), \tilde{\mu}_\Omega(y)\}$, and $\lambda_\Omega(x * y) > \beta > \max\{\lambda_\Omega(x), \lambda_\Omega(y)\}$.

It follows that $x,y \in \tilde{U}(\Omega; \tilde{t}, s)$, and $(x * y) \notin \tilde{U}(\Omega; \tilde{t}, s)$. This is a contradiction and therefore $\Omega = <\tilde{\mu}_\Omega, \lambda_\Omega>$ is a cubic AT-subalgebra of X. ⌂

Theorem 3.13. Cubic set$\Omega=(\tilde{\mu}_{\Omega},v_{\Omega})$ is a cubic AT-subalgebra of X if and only if, μ^{-}_{Ω},$\mu^{+}_{\Omega}$and v_{Ω} are fuzzy AT-subalgebras of X.

Proof. Let μ^{-}_{Ω}, μ^{+}_{Ω} and v_{Ω} be fuzzy subalgebras of X and x,y∈ X.then

$\mu^{-}_{\Omega}(x*y)\geq\min\{\mu^{-}_{\Omega}(x),\mu^{-}_{\Omega}(y)\}$,$\mu^{+}_{\Omega}(x*y)\geq\min\{\mu^{+}_{\Omega}(x),\mu^{+}_{\Omega}(y)\}$and $v_{\Omega}(x*y)\leq\max\{v_{\Omega}(x),v_{\Omega}(y)\}$.

Now,$\tilde{\mu}_{\Omega}(x*y)=[\mu^{-}_{\Omega}(x*y),\mu^{+}_{\Omega}(xy)]$
$\geq[\min\{\mu^{-}_{\Omega}(x),\mu^{-}_{\Omega}(y)\},\min\{\mu^{+}_{\Omega}(x),\mu^{+}_{\Omega}(y)\}]$
$=\text{rmin}\{[\mu^{-}_{\Omega}(x),\mu^{+}_{\Omega}(x)],[\mu^{-}_{\Omega}(y),\mu^{+}_{\Omega}(y)]\}=\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$, therefore,$\Omega$ is a cubic AT-subalgebra of X.

Conversely, assume that Ω is a cubic AT-subalgebra of X, for any x,y∈ X,

$[\mu^{-}_{\Omega}(x*y),\mu^{+}_{\Omega}(x*y)]=\tilde{\mu}_{\Omega}(x*y)\geq\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$
$=\text{rmin}\{[\mu^{-}_{\Omega}(x),\mu^{+}_{\Omega}(x)],[\mu^{-}_{\Omega}(y),\mu^{+}_{\Omega}(y)]\}$
$=[\min\{\mu^{-}_{\Omega}(x),\mu^{-}_{\Omega}(x),\{\mu^{+}_{\Omega}(y),\mu^{+}_{\Omega}(y)\}]$.

Thus$\mu^{-}_{\Omega}(x*y)\geq\{\mu^{-}_{\Omega}(x),\mu^{-}_{\Omega}(x)\}$,$\mu^{+}_{\Omega}(x*y)\geq\{\mu^{+}_{\Omega}(x),\mu^{+}_{\Omega}(x)\}$,and

$v_{\Omega}(x*y)\leq\max\{v_{\Omega}(x),v_{\Omega}(y)\}$, therefore,$\Omega$ is a cubic AT-subalgebra of X. ⌂

Theorem 3.14. Let$\Omega=(\tilde{\mu}_{\Omega},v_{\Omega})$ be a cubic AT-subalgebra of X and let n∈ ℕ (the set of natural numbers).then

(i) $\tilde{\mu}_{\Omega}(\Pi^{n}x*x)\geq\tilde{\mu}_{\Omega}(x)$ for any add number n,
(ii) $v_{\Omega}(\Pi^{n}x*x)\leq\tilde{\mu}_{\Omega}(x)$ for any add number n,
(iii) $\tilde{\mu}_{\Omega}(\Pi^{n}x*x)=\tilde{\mu}_{\Omega}(x)$ for any even number n,
(iv) $v_{\Omega}(\Pi^{n}x*x)\leq\tilde{\mu}_{\Omega}(x)$ for any even number n.

Proof. Let $x\in X$ and assum that n is odd.then n=2p-1 for some positive integer p. We prove the theorem by induction.

Now, $\tilde{\mu}_{\Omega}(x*x)=\tilde{\mu}_{\Omega}(0)\geq\tilde{\mu}_{\Omega}(x)andv_{\Omega}(x*x)=v_{\Omega}(0)\leq v_{\Omega}(x)$. Suppose that

$\tilde{\mu}_{\Omega}(\Pi^{2p-1}x*x)\geq\tilde{\mu}_{\Omega}(x)andv_{\Omega}(\Pi^{2p-1}x*x)\leq v_{\Omega}(x)$, then by assumption,

$\tilde{\mu}_{\Omega}(\Pi^{2(p+1)-1)}x*x=\tilde{\mu}_{\Omega}(\Pi^{2p+1}x*x)=\tilde{\mu}_{\Omega}(\Pi^{2p-1}x*(x*(x*x)))$
$=\tilde{\mu}_{\Omega}(\Pi^{2p-1}x*x)\geq\tilde{\mu}_{\Omega}(x)$ and
$v_{\Omega}(\Pi^{2(p+1)-1}x*x)=v_{\Omega}(\Pi^{2p+1}x*x)=v_{\Omega}(\Pi^{2p-1}x*(x(*x*x)))$
$=v_{\Omega}(\Pi^{2p-1}x*x)\geq v_{\Omega}(x)$, which proves (i)and(ii).

Proofs are similar to the cases (iii)and(iv).

The sets $\{x\in X|\tilde{\mu}_{\Omega}(x)=\tilde{\mu}_{\Omega}(0)\}$ and $\{x\in X|v_{\Omega}(x)=v_{\Omega}(0)\}$ are denoted by$I_{\tilde{\mu}_{\Omega}}$and$I_{v_{\Omega}}$ respectively. This two sets are also AT-subalgebras of X. ⌂

Theorem 3.15. Let $\Omega=(\tilde{\mu}_{\Omega},v_{\Omega})$ be a cubic AT-subalgebra of X, then the sets $I_{\tilde{\mu}_{\Omega}}$ and $I_{v_{\Omega}}$ are AT-subalgebras of X.

Proof. Let x, y∈ $I_{\tilde{\mu}_{\Omega}}$.then$\tilde{\mu}_{\Omega}(x)=\tilde{\mu}_{\Omega}(0)=\tilde{\mu}_{\Omega}(y)$and so,

$\tilde{\mu}_{\Omega}(x*y)\geq\min\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}=\tilde{\mu}_{\Omega}(0)$ by Proposition (3.3),we know that

$\tilde{\mu}_{\Omega}(x*y)=\tilde{\mu}_{\Omega}(0)$ or equivalently $x*y\in I_{\tilde{\mu}_{\Omega}}$.

Again, let x, y∈ $I_{v_{\Omega}}$.then$v_{\Omega}(x)=v_{\Omega}(0)=v_{\Omega}(y)$and so,$v_{\Omega}(x*y)\leq\max\{v_{\Omega}(x),v_{\Omega}(y)\}=v_{\Omega}(0)$.

Again by Proposition (3.3), we know that$v_{\Omega}(x*y)=v_{\Omega}(0)$ or equivalently $x*y\in I_{v_{\Omega}}$. Hence, sets $I_{\tilde{\mu}_{\Omega}}$and$I_{v_{\Omega}}$are AT-subalgebras of X. ⌂

Theorem 3.16. Let B a nonempty subset of X and $\Omega=(\widetilde{\mu_{\Omega}},v_{\Omega})$ be a cubic set of X defined by

$\tilde{\mu}_{\Omega}(x)=\begin{cases}[\alpha_1,\alpha_2], \text{if } x\in B\\ [\beta_1,\beta_2], \text{otherwise}\end{cases}$ and $v_{\Omega}(x)=\begin{cases}\gamma, \text{if } x\in B\\ \delta, \text{otherwise}\end{cases}$

For all $[\alpha_1,\alpha_2],[\beta_1,\beta_2]\in D[0,1]$ and $\gamma,\delta\in[0,1]$with$[\alpha_1,\alpha_2]\geq[\beta_1,\beta_2]and\gamma\leq\delta$.

Then Ω is a cubic AT-subalgebra of X if and only if, B an AT-subalgebra of X. Moreover, $I_{\tilde{\mu}_{\Omega}}$=B=$I_{V_{\Omega}}$.

Proof.

LetΩ be a cubic AT-subalgebra of X and x, y∈B, then
$\tilde{\mu}_{\Omega}(x*y)\geq\min\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}=\text{rmin}\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}=[\alpha_1,\alpha_2]$ and
$v_{\Omega}(x*y)\leq\max\{v_{\Omega}(x),v_{\Omega}(y)\}=\{\gamma,\gamma\}=\gamma$.

So $x*y\in B$.Hence B is an AT-subalgebra of X.

Conversely, suppose that B is AT-subalgebra of X and let x, y ∈X. Consider two cases.

Case 1 If $x,y\in B$ then $x*y\in B$, thus $\tilde{\mu}_{\Omega}(x*y)=[\alpha_1,\alpha_2]=\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and
$v_{\Omega}(x*y)=\gamma=\max\{v_{\Omega}(x),v_{\Omega}(y)\}=\max\{\gamma,\gamma\}$.

Case 2 if $x\notin B$ or $y\notin B$, then $\tilde{\mu}_{\Omega}(x*y)\geq[\beta_1,\beta_2]=\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and
$v_{\Omega}(x*y)\leq\delta=\max\{v_{\Omega}(x),v_{\Omega}(y)\}$.

Hence, Ω is cubic AT-subalgebra of X. ⌂

Now, $I_{\tilde{\mu}_{\Omega}}=\{x\in X|\tilde{\mu}_{\Omega}(x)=\tilde{\mu}_{\Omega}(0)\}=\{x\in X|\tilde{\mu}_{\Omega}(x)=[\alpha_1,\alpha_2]\}$=B and $I_{V_{\Omega}}=\{x\in X|v_{\Omega}(x)=v_{\Omega}(0)\}$=and$I_{V_{\Omega}}=\{x\in X|v_{\Omega}(x)=\gamma\}$=B.

Definition 3.17. Let $\Omega=(\tilde{\mu}_{\Omega},v_{\Omega})$be a cubic set of X. For $[s_1,s_2]\in D[0,1]$and $t\in[0,1]$,the set $U(\tilde{\mu}_{\Omega}|[s_1,s_2])=\{x\in X|\tilde{\mu}_{\Omega}(x)\geq[s_1,s_2]\}$is called upper$[s_1,s_2]$-Level of Ω and $L(v_{\Omega}|t)=\{x\in X|v_{\Omega}(x)\leq t\}$is called Lower t-Level of Ω.

Theorem 3.18. If a cubic set $\Omega=(\tilde{\mu}_{\Omega},v_{\Omega})$ is a cubic AT-subalgebra of X, then the upper
$[s_1,s_2]$-Level and Lower t-Level of Ω are ones of X.

Proof. Let x,y∈ $U(\tilde{\mu}_{\Omega}|[s_1,s_2])$, then $\tilde{\mu}_{\Omega}(x)\geq[s_1,s_2]$ and $\tilde{\mu}_{\Omega}(y)\leq[s_1,s_2]$. It follows that
$\tilde{\mu}_{\Omega}(x*y)\geq\min\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}\geq[s_1,s_2]$, so that $x*y\in U(\tilde{\mu}_{\Omega}|[s_1,s_2])$.

Hence $U(\tilde{\mu}_{\Omega}|[s_1,s_2])$ is AT-subalgebra of X. Let $x*y\in L(v_{\Omega}|t)$, then $v_{\Omega}(x)\leq t$ and $v_{\Omega}(y)\leq t$. It follows that$v_{\Omega}(x*y)\leq\max\{v_{\Omega}(x),v_{\Omega}(y)\}\leq t$, so that $x*y\in L(v_{\Omega}|t)$.

Hence, L $(v_{\Omega}|t)$is subalgebra of X. ⌂

Corollary 3.19. Let $\Omega=(\tilde{\mu}_{\Omega},v_{\Omega})$ be a cubic AT-subalgebra of X, then
$\Omega([s_1,s_2];t)=U(\tilde{\mu}_{\Omega}|[s_1,s_2])L(v_{\Omega}|t)=\{x\in X|\tilde{\mu}_{\Omega}(x)\geq[s_1,s_2],v_{\Omega}(x)\leq t\}$ is a cubic AT-subalgebra of X

The following example shows that the converse of Corollary (3.19) is not valid

Example 3.20. Let X={0,a,b,c,d,e}be AT-algebra and cubic set $\Omega=(\tilde{\mu}_{\Omega},v_{\Omega})$ of X by

$$\tilde{\mu}_{\Omega}(x)=\begin{cases}[0.6,0.8], \text{if } x=0,\\ [0.5,0.6], \text{if } x\in\{a,b,c\},\\ [0.3,0.4], \text{if } x\in\{d,e\},\end{cases} \text{ and } v_{\Omega}(x)=\begin{cases}0.1, \text{if } x=0,\\ 0.3, \text{if } x\in\{a,b,c\},\\ 0.8, \text{if } x\in\{d,e\},\end{cases}$$

We take $[s_1,s_2]=[0.41,0.48]$and t=0.4, then $\Omega([s_1,s_2];t)=U(\tilde{\mu}_{\Omega}|[s_1,s_2])L(v_{\Omega}|t)=\{x\in X|\tilde{\mu}_{\Omega}(x)\geq[s_1,s_2],v_{\Omega}(x)\leq t\}$
={a,b,c}∩{0,a,b,d}={0,a,b} is AT-subalgebra of X, but
$\Omega=(\tilde{\mu}_{\Omega},v_{\Omega})$is not a cubic AT-subalgebra since $\tilde{\mu}_{\Omega}(1*3)\ngeq\text{rmin}\{\tilde{\mu}_{\Omega}(1),\tilde{\mu}_{\Omega}(3)\}$ and
$v_{\Omega}(2*4\nleq)\max\{v_{\Omega}(2),v_{\Omega}(4)\}$.

4. Cubic AT-ideals of AT-algebras

In this section, we will introduce a new notion called cubic

AT-ideal of AT-algebras and study several properties of it.

Definition 4.1. Let (X ;*, 0)be an AT-algebra. A cubic set Ω =<$\tilde{\mu}_\Omega$ (x),λ_Ω(x)>of X is called cubic AT-ideal of X if, for all x, y, z ∈X:

(A_1) $\tilde{\mu}_\Omega$ (0) ≽$\tilde{\mu}_\Omega$ (x), and λ_Ω (0) ≤λ_Ω (x)

(A_2) $\tilde{\mu}_\Omega$ (x∗z) ≽ rmin{$\tilde{\mu}_\Omega$ (x∗(y*z)), $\tilde{\mu}_\Omega$ (y)}, and λ_Ω (z∗x) ≤ max{λ_Ω (x∗(y*z)), λ_Ω (y)}.

Example 4.2. Let X = {0,1,2,3} in which the operation as in example (∗) be define by the following table:

*	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then (X;∗,0) is an AT-algebra. Define a cubic set Ω =<$\tilde{\mu}_\Omega$, λ_Ω> of X as follows:

fuzzy subset μ: X→ [0,1] by: $\tilde{\mu}_\Omega$(x) = $\begin{cases}[0.3,0.9] & if\ x=\{0,1\}\\ [0.1,0.6] & otherwise\end{cases}$

and λ_Ω=$\begin{cases}0.1\ if\ x\in\{0,1\}\\ 0.6\ \text{otherwise}\end{cases}$.The cubic set Ω =<$\tilde{\mu}_\Omega$, λ_Ω> is a cubic AT-ideal of X.

Proposition4.3. Let Ω =<$\tilde{\mu}_\Omega$, λ_Ω>be a cubic AT-ideal of an AT-algebra (X ;*, 0), if there exist a sequence{x_n} in X such that $\lim_{n\to\infty}\tilde{\mu}_\Omega(x_n)$ = [1,1], then $\tilde{\mu}_\Omega$(0) = [1, 1].

Proof. By definition (3.1), we have $\tilde{\mu}_\Omega$(0) ≽$\tilde{\mu}_\Omega$(x), for all x∈X. Then$\tilde{\mu}_\Omega$ (0) ≽$\tilde{\mu}_\Omega$(x_n), for every positive integer n.

Consider the inequality [1,1] ≽$\tilde{\mu}_\Omega$ (0) ≽ $\lim_{n\to\infty}\tilde{\mu}_\Omega(x_n)$ = [1,1].Hence $\tilde{\mu}_\Omega$ (0) = [1,1]. ⌂

Theorem 4.4. Let (X ;*, 0) be an AT-algebra and A cubic set Ω =<$\tilde{\mu}_\Omega$, λ_Ω>of X. A cubic set Ω of X is a cubic AT-ideal of X ifand only if, μ_A^-,$\mu_A^+$and $\lambda_\Omega$are cubic AT-ideals of X.

Proof.

Suppose that Ω is a cubic AT-ideal of X. For all x, y, z ∈X, we have

[μ_A^- (x∗z), μ_A^+ (x∗z)]=$\tilde{\mu}_\Omega$ (x∗z)≽rmin{$\tilde{\mu}_\Omega$(x∗(y*z)),$\tilde{\mu}_\Omega$ (y)}

=rmin{[μ_A^-(x∗(y∗z)),μ_A^+(x∗(y∗z))], [μ_A^- (y), μ_A^+ (y)]}

= [min{μ_A^-(x∗(y∗z)),μ_A^- (y)}, min {μ_A^+(x∗(y∗z)),μ_A^+ (y)}].

Therefore, μ_A^-(x∗z) ≥ min{μ_A^-(x∗(y*z)),μ_A^-(y)} and

μ_A^+(x∗z) ≥ min{μ_A^+(x∗(y∗z)),μ_A^+(y)}.

Similarly, we can show that λ_Ω (x∗z)≤max{λ_Ω(x∗(y∗z)),λ_Ω (y)}

Conversely, If $\mu_A^-$and $\mu_A^+$are cubic AT-ideals of X. For any x, y, z ∈X. Observe

$\tilde{\mu}_\Omega$ (x∗z) = [μ_A^- (x*z),μ_A^+ (x∗z)]

≽ [min {μ_A^- (x∗(y*z)), μ_A^- (y)},min {μ_A^+ (x∗(y*z)), μ_A^+ (y)}]

= rmin {[μ_A^- (x∗(y*z)μ_A^+ (x∗(y*z))], [μ_A^- (y), μ_A^+ (y)]}

= rmin {$\tilde{\mu}_\Omega$ (x∗(y*z)),$\tilde{\mu}_\Omega$ (y)].

Similarly, we can show that λ_Ω (x∗z) ≤ max {[λ_Ω (x∗(y*z)),λ_Ω (y)]}.

From what was mentioned above we can conclude that Ω is a cubic AT-ideal of X

Hence, we get that μ_A^-, $\mu_A^+$and $\lambda_\Omega$are cubic AT-ideals of X. ⌂

Theorem 4.5. Let {Ω_i |i∈Λ} be family of cubic AT-ideals of an AT-algebra (X ;*, 0). Then$\bigcap_{i\in\Lambda}\tilde{\mu}_{\Omega i}$is a cubic AT-ideal of X.

Proof. Let {Ω_i |i∈Λ} be family of cubic AT-ideals of X, then for any x, y, z ∈X,

(∩$\tilde{\mu}_{\Omega i}$)(0) =rinf ($\tilde{\mu}_{\Omega i}$ (0))≽ rinf ($\tilde{\mu}_{\Omega i}$ (x)) = (∩$\tilde{\mu}_{\Omega i}$)(x)

(∩$\tilde{\mu}_{\Omega i}$(x∗z)) = rinf ($\tilde{\mu}_{\Omega i}$ (x∗z))≽rinf (rmin{ $\tilde{\mu}_{\Omega i}$ (x∗(y*z)), $\tilde{\mu}_{\Omega i}$ (y)})

=rmin { rinf ($\tilde{\mu}_{\Omega i}$ (x∗(y*z)),rinf ($\tilde{\mu}_{\Omega i}$ (y))}=rmin { (∩$\tilde{\mu}_{\Omega i}$)(x∗(y*z)),(∩$\tilde{\mu}_{\Omega i}$) (y)}

Also,(∪$\lambda_{\Omega i}$)(0) = sup($\lambda_{\Omega i}$ (0))≤ sup ($\lambda_{\Omega i}$ (x)) = (∪$\lambda_{\Omega i}$)(x)

(∪$\lambda_{\Omega i}$(x∗z)) = sup ($\lambda_{\Omega i}$ (x∗z))≤ sup (max { $\lambda_{\Omega i}$ (x∗(y*z)), $\lambda_{\Omega i}$ (y)})

= max {sup($\lambda_{\Omega i}$ (x∗(y*z)),sup($\lambda_{\Omega i}$ (y))}= max { (∪$\lambda_{\Omega i}$)(x∗(y*z)), (∪$\lambda_{\Omega i}$)(y) }. ⌂

Theorem 4.6. Let (X ;*, 0) be an AT-algebra. A cubic subset Ω =<$\tilde{\mu}_\Omega$, λ_Ω> of X, then Ω is a cubic AT-ideal of X ifand only if, for all $\tilde{t}$ ∈ D[0, 1] and s ∈ [0, 1], the set $\tilde{U}$ (Ω; $\tilde{t}$,s) is an AT-ideal of X, where $\tilde{U}$ (Ω; $\tilde{t}$,s) ={ x∈X|$\tilde{\mu}_\Omega$ (x)≽ $\tilde{t}$,λ_Ω (x) ≤ s}.

Proof.

Assume that Ω =<$\tilde{\mu}_\Omega$, λ_Ω>is a cubic AT-ideal of X and let $\tilde{t}$ ∈ D[0, 1] and s ∈ [0, 1],be such that $\tilde{U}$ (Ω; $\tilde{t}$,s) ≠∅, and let x, y, z ∈ X such that (x∗(y∗z)),y∈ $\tilde{U}$ (Ω; $\tilde{t}$,s), then $\tilde{\mu}_\Omega$(x∗(y*z))≽ $\tilde{t}$, $\tilde{\mu}_\Omega$ (y)≽ $\tilde{t}$ and λ_Ω(x∗(y∗z))≤ s, λ_Ω (y) ≤ s. By (A_2), we get

$\tilde{\mu}_\Omega$ (x∗z) ≽rmin {$\tilde{\mu}_\Omega$(x∗(y∗z)),$\tilde{\mu}_\Omega$ (y) } ≽ $\tilde{t}$, and

λ_Ω (x∗z) ≤ max {λ_Ω(x∗(y∗z)),λ_Ω (y) } ≤ s.

Hence the set $\tilde{U}$ (Ω; $\tilde{t}$, s) is anAT-ideal of X.

Conversely, suppose that $\tilde{U}$ (Ω; $\tilde{t}$,s) is an AT-ideal of X and let x, y, z ∈ X be such that$\tilde{\mu}_\Omega$ (x∗z)≺rmin {$\tilde{\mu}_\Omega$(x∗(y*z)),$\tilde{\mu}_\Omega$ (y)}, andλ_Ω (x∗z) > max {λ_Ω(x∗(y*z)),λ_Ω (y)}.

Consider $\tilde{\beta}$ =1/2 { $\tilde{\mu}_\Omega$(x∗z) + rmin{$\tilde{\mu}_\Omega$(x∗(y∗z))), $\tilde{\mu}_\Omega$(y)} } and B= 1/2 { λ_Ω (x∗z) + max{λ_Ω (x∗(y∗z))), λ_Ω (y)}}.

We have $\tilde{\beta}$ ∈ D[0, 1] and B∈ [0, 1], and $\tilde{\mu}_\Omega$ (x∗z) ≺ $\tilde{\beta}$ ≺rmin {$\tilde{\mu}_\Omega$(x∗(y*z)),$\tilde{\mu}_\Omega$ (y)}, and λ_Ω (x∗z) >B> max {λ_Ω(x∗(y∗z)),λ_Ω (y)}.

It follows that (x∗(y*z)),y∈ $\tilde{U}$ (Ω; $\tilde{t}$,s), and (x∗z)∉ $\tilde{U}$ (Ω; $\tilde{t}$,s). This is a contradiction and therefore Ω =<$\tilde{\mu}_\Omega$, λ_Ω> is a cubic AT-ideal of X. ⌂

Proposition4.7. If Ω =<$\tilde{\mu}_\Omega$, λ_Ω>is a cubic AT-ideal of AT-algebra X, then

$\tilde{\mu}_{\Omega}$ (x ∗ (x ∗ y) ≽ $\tilde{\mu}_{\Omega}$ (y), and λ_{Ω}(x ∗ (x ∗ y)) ≤ λ_{Ω}(y).

Proof. Taking z=x ∗ $y$in(ATI_2)and using (AT_3)in (ATI_1),we

$\tilde{\mu}_{\Omega}$(x ∗ (x ∗ y) ≽ rmin{$\tilde{\mu}_{\Omega}$(x∗(y∗(x∗y))),$\tilde{\mu}_{\Omega}$(y) }=rmin{$\tilde{\mu}_{\Omega}$(x∗(x∗(y∗y))),$\tilde{\mu}_{\Omega}$(y)} =rmin{$\tilde{\mu}_{\Omega}$(x∗(x∗0)),$\tilde{\mu}_{\Omega}$(y)}=rmin{$\tilde{\mu}_{\Omega}$(0),$\tilde{\mu}_{\Omega}$(y)}=$\tilde{\mu}_{\Omega}$(y),

λ_{Ω}(x∗(x∗y))≤ max {λ_{Ω}(x∗(y∗(x∗y))),λ_{Ω}(y)} =max{λ_{Ω}(x∗(x∗(y∗y))),λ_{Ω}(y)}=max{λ_{Ω}(x∗(x∗0)),λ_{Ω}(y)} =max{λ_{Ω}(0),λ_{Ω}(y)}=λ_{Ω}(y). ⌂

5. Homomorphism of Cubic AT-ideal (AT-subalgebra) of AT-algebras

In this section, we will present some results on images and preimages of cubicAT-ideals of AT-algebras.

Definition 5.1[3].

Let f : (X; ∗ ,0) →(Y; ∗ ',0') be a mapping from the set X to a set Y. If Ω =<$\tilde{\mu}_{\Omega}$, λ_{Ω}>is a cubic subset of X, then the cubic subset β =<$\tilde{\mu}_{\beta}$, λ_{β}>of Y defined by:

$$f(\tilde{\mu}_{\Omega})(y)=\tilde{\mu}_{\beta}(y)=\begin{cases} r\sup_{x\in f^{-1}(y)} \tilde{\mu}_{\Omega}(x) & if\ f^{-1}(y)=\{x\in X, f(x)=y\}\neq\phi \\ 0 & otherwise \end{cases}$$

$$f(\lambda_{\Omega})(y)=\lambda_{\beta}(y)=\begin{cases} \inf_{x\in f^{-1}(y)} \lambda_{\Omega}(x) & if\ f^{-1}(y)=\{x\in X, f(x)=y\}\neq\phi \\ 1 & otherwise \end{cases}$$

is said to be the image of Ω under f .

Similarly if β =<$\tilde{\mu}_{\beta}$, λ_{β}>is a cubic subset of Y, then the cubic subset Ω = (β ∘ f) in X (i.e the cubic subset defined by $\tilde{\mu}_{\Omega}$ (x) = $\tilde{\mu}_{\beta}$ (f (x)), λ_{Ω} (x) = λ_{β} (f (x)) for all x ∈X) is called the preimage of βunder f).

Theorem 5.2. An onto homomorphic preimage of cubic AT-subalgebra is also cubic AT-subalgebra.

Proof. Let f : (X; ∗ ,0) →(Y; ∗ ',0') be onto homomorphism from an AT-algebra X into an AT-algebra Y.

If β =<$\tilde{\mu}_{\beta}$, λ_{β}>is a cubic AT-subalgebra of Y and Ω =<$\tilde{\mu}_{\Omega}$, λ_{Ω}> the preimage of β under f , then $\tilde{\mu}_{\Omega}$ (x) = $\tilde{\mu}_{\beta}$ (f (x)), λ_{Ω} (x) = λ_{β} (f (x)), for all x ∈X.

Let x ∈X, then

($\tilde{\mu}_{\Omega}$)(0) = $\tilde{\mu}_{\beta}$ (f (0))≽$\tilde{\mu}_{\beta}$ (f (x)) = $\tilde{\mu}_{\Omega}$ (x), and (λ_{Ω})(0) = λ_{β} (f (0))≤λ_{β} (f (x)) = λ_{Ω} (x).

Now, let x, y ∈ X, then

$\tilde{\mu}_{\Omega}$ (x ∗ y) = $\tilde{\mu}_{\beta}$(f (x ∗ y))≽rmin {$\tilde{\mu}_{\beta}$(f (x),$\tilde{\mu}_{\beta}$ (f (y)) } = rmin {$\tilde{\mu}_{\Omega}$ (x),$\tilde{\mu}_{\Omega}$ (y)}, and

λ_{Ω} (x ∗ y) = λ_{β}(f (x ∗ y)) ≤ max {λ_{β} (f (x),λ_{β} (f (y))} = max {λ_{Ω} (x),λ_{Ω} (y)}. ⌂

Definition 5.3. Let f : (X; ∗ ,0) →(Y; ∗ ',0') be a mapping from a set X into a set Y.

Ω =<$\tilde{\mu}_{\Omega}$, λ_{Ω}>is a cubic subset of X has sup and inf properties if for any subset T of X, there exist t, s ∈ T such that $\tilde{\mu}_{\Omega}(t)= r\sup_{t\in T}\tilde{\mu}_{\Omega}(t)$ and $\lambda_{\Omega}(s)=\inf_{s\in T}\lambda_{\Omega}(s)$.

Theorem 5.4. Let f : (X; ∗ ,0) →(Y; ∗ ',0') be a homomorphism from an AT-algebra X into an AT-algebra Y. For every cubic AT-subalgebra Ω =<$\tilde{\mu}_{\Omega}$, λ_{Ω}> of X, then f (Ω) is a cubic AT-subalgebra of Y.

Proof. By definition

$\tilde{\mu}_{\beta}(y')=f(\tilde{\mu}_{\Omega})(y')= r\sup_{t\in f^{-1}(y')}\tilde{\mu}_{\Omega}(x)$ and

$\lambda_{\beta}(y')= f(\lambda_{\Omega})(y')= \inf_{t\in f^{-1}(y')}\lambda_{\Omega}(x)$ for all y' ∈ Y and

rsup(∅) = [0, 0] and inf (∅) = 0. We have prove that

$\tilde{\mu}_{\Omega}$ (x' ∗ y') ≽rmin {$\tilde{\mu}_{\Omega}$ (x'), $\tilde{\mu}_{\Omega}$ (y')}, and λ_{Ω} (x' ∗ y')≤ max{λ_{Ω} (x'), λ_{Ω} (y')}, for all x', y' ∈ Y.

Let f : (X; ∗ ,0) →(Y; ∗ ',0') be a homomorphism of AT-algebras,

Ω =<$\tilde{\mu}_{\Omega}$, λ_{Ω}> is a cubic AT-subalgebra of X has sup and inf properties and

β =<$\tilde{\mu}_{\beta}$, λ_{β}>the image of Ω =<$\tilde{\mu}_{\Omega}$, λ_{β}> under f .

Since Ω =<$\tilde{\mu}_{\Omega}$, λ_{Ω}> is a cubic AT-subalgebra of X, we have ($\tilde{\mu}_{\Omega}$)(0)≽$\tilde{\mu}_{\Omega}$ (x), and (λ_{Ω})(0)≤λ_{Ω} (x), for all x ∈X.

Note that, 0 ∈ f^{-1} (0') where 0,0' are the zero of X and Y, respectively. Thus

$$\tilde{\mu}_{\beta}(0')= r\sup_{t\in f^{-1}(0')}\tilde{\mu}_{\Omega}(t)=\tilde{\mu}_{\Omega}(0) \succcurlyeq \tilde{\mu}_{\Omega}(x) = r\sup_{t\in f^{-1}(x')}\tilde{\mu}_{\Omega}(t)=\tilde{\mu}_{\beta}(x'),$$

$$\lambda_{\beta}(0')= \inf_{t\in f^{-1}(0')}\lambda_{\Omega}(t) = \lambda_{\Omega}(0)\leq \lambda_{\Omega}(x) = \inf_{t\in f^{-1}(x')}\lambda_{\Omega}(t) = \lambda_{\beta}(x'),$$

for all x ∈X, which implies that $\tilde{\mu}_{\beta}$ (0') ≽ $\tilde{\mu}_{\beta}(x')$,and $\lambda_{\beta}(0')\leq \lambda_{\beta}(x')$, for all x' ∈Y.

For any x', y' ∈ Y, let x_0∈f^{-1} (x') and y_0∈f^{-1} (y') be such that

$$\tilde{\mu}_{\Omega}(x_0)= r\sup_{t\in f^{-1}(x')}\tilde{\mu}_{\Omega}(t),\quad \tilde{\mu}_{\Omega}(y_0)= r\sup_{t\in f^{-1}(y')}\tilde{\mu}_{\Omega}(t)$$ and

$\tilde{\mu}_{\Omega}(x_0 * y_0)$ = $\tilde{\mu}_{\beta}\{f(x_0 * y_0)\}$ = $\tilde{\mu}_{\beta}(x'*y')$ $= r\sup_{(x_0 * z_0)\in f^{-1}(x'*y')} \tilde{\mu}_{\Omega}(x_0 * y_0)$ $= r\sup_{t\in f^{-1}(x'*y')} \tilde{\mu}_{\Omega}(t)$. Also,

$\lambda_{\Omega}(x_0) = \inf_{t\in f^{-1}(x')} \lambda_{\Omega}(t)$, $\lambda_{\Omega}(y_0) = \inf_{t\in f^{-1}(y')} \lambda_{\Omega}(t)$ and

$\lambda_{\Omega}(x_0 * y_0) = \lambda_{\beta}\{f(x_0 * y_0)\} = \lambda_{\beta}\{f(x'*y')\} = \inf_{(x_0 * y_0)\in f^{-1}(x'*y')} \lambda_{\Omega}(x_0 * y_0)$

$= \inf_{t\in f^{-1}(x'*z')} \lambda_{\Omega}(t)$. Then

$\tilde{\mu}_{\beta}(x'*y')$ $= r\sup_{t\in f^{-1}(x'*y')} \tilde{\mu}_{\Omega}(t)$ = $\tilde{\mu}_{\Omega}(x_0 * y_0)$ ⩾rmin { $\tilde{\mu}_{\Omega}(x_0)$, $\tilde{\mu}_{\Omega}(y_0)$},

$= rmin\{$ $r\sup_{t\in f^{-1}(x')} \tilde{\mu}_{\Omega}(t)$, $r\sup_{t\in f^{-1}(y')} \tilde{\mu}_{\Omega}(t)$ } = rmin { $\tilde{\mu}_{\beta}(x')$, $\tilde{\mu}_{\beta}(y')$ }and

$\lambda_{\beta}(x'*y') = \inf_{t\in f^{-1}(x'*y')} \lambda\Omega_{\Omega}(t) = \lambda_{\Omega}(x'*y')$

≤ max { $\lambda_{\Omega}(x_0)$, $\lambda_{\Omega}(y_0)$ } = max { $\inf_{t\in f^{-1}(x')} \lambda_{\Omega}(t)$, $\inf_{t\in f^{-1}(y')} \lambda_{\Omega}(t)$ }

Hence, β is a cubic AT-subalgebra of Y. ⌂

Theorem 5.5.
Let $\Omega=(\tilde{\mu}_{\Omega},\nu_{\Omega})$ be a cubic set of X such that the sets $U(\tilde{\mu}_{\Omega}|[s_1,s_2])$ and $L(\upsilon_{\Omega}|t)$ are AT-subalgebras of X for every $[s_1,s_2]\in D[0,1]$ and $t\in[0,1]$, then $\Omega=(\tilde{\mu}_{\Omega},\nu_{\Omega})$ is a cubic AT-subalgebra of X.

Proof.
Let $U(\tilde{\mu}_{\Omega}|[s_1,s_2])$ and $L(\nu_{\Omega}|t)$ are AT-subalgebras of X, for every $[s_1,s_2]\in D[0,1]$
and $t\in[0,1]$. on the contrary, let $x_0, y_0 \in X$ be such that $\tilde{\mu}_{\Omega}(x_0, y_0) < \text{rmin}\{\tilde{\mu}_{\Omega}(x_0), \tilde{\mu}_{\Omega}(y_0)\}$.
Let $\tilde{\mu}_{\Omega}(x_0)=[\theta_1,\theta_2]$ and $\tilde{\mu}_{\Omega}(y_0)=[\theta_3,\theta_4]$ and $\tilde{\mu}_{\Omega}(x_0, y_0)=[s_1,s_2]$.
Then $[s_1,s_2]<\text{rmin}\{=[\theta_1,\theta_2],[\theta_3,\theta_4]\}=[\min\{\theta_1,\theta_2\},\min\{\theta_3,\theta_4\}]$. So, $s_1<\min\{\theta_1,\theta_3\}$ and $S_2<\{\theta_2,\theta_4\}$. Let us cosider,
$[\rho_1,\rho_2]= \frac{1}{2}[\tilde{\mu}_{\Omega}(x_0 * y_0)+ \text{rmin}\{\tilde{\mu}_{\Omega}(x_0), \tilde{\mu}_{\Omega}(y_0)\}] = \frac{1}{2}\left[[s_1,s_2]+[min\{\theta_1,\theta_3\}, min\{\theta_2,\theta_4\}]\right]$
$=\left[\frac{1}{2}(s_1+\min\{\theta_1,\theta_3\}), \frac{1}{2}(s_2+\min\{\theta_2,\theta_4\})\right]$.
Therefore, $\min\{\theta_1,\theta_3\}>\rho_1=\frac{1}{2}(s_1+\min\{\theta_1,\theta_3\})>s_1$ and
$\min\{\theta_2,\theta_4\}>\rho_2=\frac{1}{2}(s_2+\min\{\theta_2,\theta_4\})>s_2$.
Hence $[\min\{\theta_1,\theta_3\},\min\{\theta_2,\theta_4\}]>[\rho_1,\rho_2]>[s_1,s_2]$, so that $(x_0 * y_0)\notin U(\tilde{\mu}_{\Omega}|[s_1,s_2])$ which is a contradiction since $\tilde{\mu}_{\Omega}(x_0)=[\theta_1,\theta_2]>[\min\{\theta_1,\theta_3\},\min\{\theta_2,\theta_4\}]> [\rho_1,\rho_2]$ and $\tilde{\mu}_{\Omega}(y_0)=[\theta_2,\theta_3]>[\min\{\theta_1,\theta_3\},\min\{\theta_2,\theta_4\}]> [\rho_1,\rho_2]$ this implies
$(x_0 * y_0)\in U(\tilde{\mu}_{\Omega}|[s_1,s_2])$. Thus $\tilde{\mu}_{\Omega}(x* y) \geq \text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$, for all $x, y\in X$.
Again, Let $x_0, y_0 \in X$ such that $\nu(x_0, y_0)>\max\{\nu_{\Omega}(x_0), \nu_{\Omega}(y_0)\}$. Let $\nu(x_0)=\eta_1$, $\nu_{\Omega}(y_0)=\eta_2$ and $\nu_{\Omega}(x_0 * y_0) = t$. Then $t>\max\{\eta_1,\eta_2\}$.
Let us consider, $t_1=\frac{1}{2}[\nu_{\Omega}(x_0 * y_0)+\max\{\nu_{\Omega}(x_0), \nu(y_0)\}]$.
We get that $t_1=\frac{1}{2}(t_1+\max\{\eta_1,\eta_2\})$, therefore,
$\eta_1<t_1=\frac{1}{2}(t_1+\max\{\eta_1,\eta_2\})<t$ and $\eta_2<t_1==\frac{1}{2}(t+\max\{\eta_1,\eta_2\})<t$, hence, $\max\{\eta_1,\eta_2\}<t_1<t=\nu_{\Omega}(x_0 * y_0)$. So that $x_0 * y_0 \notin L(\upsilon_{\Omega}|t)$ which is a contradiction since $\nu_{\Omega}(x_0) = \eta_1\leq\max\{\eta_1,\eta_2\}<t_1$ and $\nu(y_0)=\eta_2\leq\max\{\eta_1,\eta_2\}<t_1$, this implies $x_0, y_0\in L(\nu_{\Omega}|t)$ this implies $\nu_{\Omega}(x* y)\leq \max\{\nu_{\Omega}(x), \nu_{\Omega}(y)\}$, for all $x, y\in X$. ⌂

Theorem 5.6. Any AT-subalgebra of X can be realized as both the upper$[s_1,s_2]$-Level and Lower t-Level of some cubic AT-subalgebra of X.

Proof. Let P be a cubic AT-subalgebra of X and Ω be cubic set on X defined by
Let P be a cubic AT-subalgebra of X and Ω be cubic set on X defined by
$\tilde{\mu}_{\Omega}(x)=\begin{cases}[\alpha_1,\alpha_2], \text{if } x\in P\\ [0,0], \text{otherwise}\end{cases}$ and $\nu_{\Omega}(x)=\begin{cases}\beta, \text{if } x\in P\\ 1, \text{othrwise}\end{cases}$
For all $[\alpha_1,\alpha_2]\in D[0,1]$ and $\beta\in[0,1]$, we consider the following cases:
***Case* 1)** if x, y∈ P, then $\tilde{\mu}_{\Omega}(x)=[\alpha_1,\alpha_2]$, $\nu_{\Omega}(x)=\beta$ and $\tilde{\mu}_{\Omega}(y)=[\alpha_1,\alpha_2]$, $\nu_{\Omega}(y)=\beta$.
Thus, $\tilde{\mu}_{\Omega}(x* y)=[\alpha_1,\alpha_2]=\text{rmin}\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}=\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and
$\nu_{\Omega}(x* y)=\beta=\max[\beta_1,\beta_2]=\max\{\nu_{\Omega}(x),\nu_{\Omega}(y)\}$.
Case 2 **)**if x ∈ Pand y∉ P,then $\tilde{\mu}_{\Omega}(x)=[\alpha_1,\alpha_2]$, $\nu_{\Omega}(x)=\beta$ and $\tilde{\mu}_{\Omega}(y)=[0,0]$, $\nu_{\Omega}(y)=1$.
Thus $\tilde{\mu}_{\Omega}(x* y)=[0,0]\geq\text{rmin}\{[\alpha_1,\alpha_2],[0,0]\}=\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and $\nu_{\Omega}(x* y)\leq 1=\max[\beta_1,1]=\max\{\nu_{\Omega}(x),\nu_{\Omega}(y)\}$.
***Case3*)** if x∉ P and y∈ P, then $\tilde{\mu}_{\Omega}(x)=[0,0]$, $\nu_{\Omega}(x)=1$ and $\tilde{\mu}_{\Omega}(y)=[\alpha_1,\alpha_2]$, $\nu_{\Omega}(y)=\beta$
Thus, $\tilde{\mu}_{\Omega}(x* y)=[0,0]\geq\text{rmin}\{[0,0],[\alpha_1,\alpha_2]\}=\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and
$\nu_{\Omega}(x* y)\leq 1+ =\max[1,\beta_1]=\max\{\nu_{\Omega}(x),\nu_{\Omega}(y)\}$.
Case4) x∉ P, y ∉ P and y, then $\tilde{\mu}_{\Omega}(x)=[0,0]$, $\nu_{\Omega}(x)=1$ and $\tilde{\mu}_{\Omega}(y)=[0,0]$, $\nu_{\Omega}(y)=1$
Now, $\tilde{\mu}_{\Omega}(x* y)=[0,0]=\text{rmin}\{[0,0],[0,0]\}=\text{rmin}\{\tilde{\mu}_{\Omega}(x),\tilde{\mu}_{\Omega}(y)\}$ and $\nu_{\Omega}(x* y)\leq 1=\max[1,1]=\max\{\nu_{\Omega}(x),\nu_{\Omega}(y)\}$.
Therefore, Ω is a cubic AT-subalgebra of X.⌂

Theorem 5.7. An onto homomorphic preimage of cubic AT-ideal is also cubic AT-ideal.

Proof.
Let f : (X;∗,0) →(Y;∗',0') be onto homomorphism from an AT-algebra X into an AT-algebra Y.

If β =<$\tilde{\mu}_\beta$, λ_β>is a cubic AT-ideal of Y and Ω =<$\tilde{\mu}_\Omega$, λ_Ω>the preimage of β under f, then $\tilde{\mu}_\Omega$ (x) = $\tilde{\mu}_\beta$ (f (x)), λ_Ω (x) = λ_β (f (x)), for all x ∈X. Let x ∈X, then ($\tilde{\mu}_\Omega$)(0) = $\tilde{\mu}_\beta$ (f (0))≽$\tilde{\mu}_\beta$ (f (x)) = $\tilde{\mu}_\Omega$ (x), and (λ_Ω)(0) = λ_β (f (0))≤λ_β (f (x)) = λ_Ω (x).

Now, let x, y, z ∈ X, then

$\tilde{\mu}_\Omega$ (x∗z) = $\tilde{\mu}_\beta$ (f (x∗z))≽rmin {$\tilde{\mu}_\beta$ (f (x∗(y*z)),$\tilde{\mu}_\beta$ (f (y)) }

=rmin {$\tilde{\mu}_\Omega$(x∗(y*z)),$\tilde{\mu}_\Omega$ (y)}, and

λ_Ω (x∗z) = λ_β (f (z∗x))≤ max {λ_β (f (x∗(y∗z)),λ_β (f (y))}

= max {λ_Ω(x∗(y∗z)),λ_Ω (y)}. ⌂

Definition 5.8. Let f : (X;∗,0) →(Y;∗',0') be a mapping from a set X into a set Y. Ω =<$\tilde{\mu}_\Omega$, λ_Ω>is a cubic subset of X has sup and inf properties if for any subset T of X, there exist t, s ∈ T such that $\tilde{\mu}_\Omega(t) = r\sup_{t\in T}\tilde{\mu}_\Omega(t)$ and $\lambda_\Omega(s) = \inf_{s\in T}\lambda_\Omega(s)$.

Theorem 5.9. Let f : (X;∗,0) →(Y;∗',0') be a homomorphism from an AT-algebra X into an AT-algebra Y. For every cubic AT-ideal Ω =<$\tilde{\mu}_\Omega$, λ_Ω> of X, then f (Ω) is a cubic AT-ideal of Y.

Proof. By definition $\tilde{\mu}_\beta(y') = f(\tilde{\mu}_\Omega)(y') = r\sup_{t\in f^{-1}(y')}\tilde{\mu}_\Omega(x)$ and $\lambda_\beta(y') = f(\lambda_\Omega)(y') = \inf_{t\in f^{-1}(y')}\lambda_\Omega(x)$ for all y' ∈ Y and rsup(∅) = [0, 0] and inf (∅) = 0. We have prove that $\tilde{\mu}_\Omega$ (x'∗z') ≽rmin {$\tilde{\mu}_\Omega$ (x' ∗(y'*z')), $\tilde{\mu}_\Omega$ (y')}, and λ_Ω (x'∗z')≤ max{λ_Ω (x' ∗(y'*z')), λ_Ω (y')}, for all x', y', z' ∈ Y.

Let f : (X;∗,0) →(Y;∗',0') be a homomorphism of AT-algebras, Ω =<$\tilde{\mu}_\Omega$, λ_Ω> is a cubic AT-idealof X has sup and inf properties and β =<$\tilde{\mu}_\beta$, λ_β>the image of Ω =<$\tilde{\mu}_\Omega$, λ_β>under f.

Since Ω =<$\tilde{\mu}_\Omega$, λ_Ω> is a cubic AT-ideal of X, we have($\tilde{\mu}_\Omega$)(0) ≽$\tilde{\mu}_\Omega$ (x), and (λ_Ω)(0)≤λ_Ω (x), for all x ∈X.

Note that, 0 ∈ f^{-1} (0') where 0,0' are the zero of X and Y, respectively. Thus

$\tilde{\mu}_\beta(0') = r\sup_{t\in f^{-1}(0')}\tilde{\mu}_\Omega(t) = \tilde{\mu}_\Omega(0)$ ≽$\tilde{\mu}_\Omega$(x) = $r\sup_{t\in f^{-1}(x')}\tilde{\mu}_\Omega(t) = \tilde{\mu}_\beta(x')$,

$\lambda_\beta(0') = \inf_{t\in f^{-1}(0')}\lambda_\Omega(t) = \lambda_\Omega(0) \le \lambda_\Omega(x)$ = $\inf_{t\in f^{-1}(x')}\lambda_\Omega(t) = \lambda_\beta(x')$, for all x ∈X, which implies that $\tilde{\mu}_\beta$(0') ≽ $\tilde{\mu}_\beta(x')$, and $\lambda_\beta(0') \le \lambda_\beta(x')$, for all x' ∈Y.

For any x', y', z' ∈ Y, let $x_0 \in f^{-1}$(x'), $y_0 \in f^{-1}$(y'), and $z_0 \in f^{-1}$(z')be such that

$\tilde{\mu}_\Omega(x_0 * (y_0 * z_0)) = r\sup_{t\in f^{-1}(x'*(y'*z'))}\tilde{\mu}_\Omega(t)$,

$\tilde{\mu}_\Omega(y_0) = r\sup_{t\in f^{-1}(y')}\tilde{\mu}_\Omega(t)$ and

$\tilde{\mu}_\Omega(x_0 * z_0)$ = $\tilde{\mu}_\beta\{f(x_0 * z_0)\}$ = $\tilde{\mu}_\beta(x'*z')$ = $r\sup_{(x_0*z_0)\in f^{-1}(x'*z')}\tilde{\mu}_\Omega(x_0 * z_0)$

= $r\sup_{t\in f^{-1}(x'*z')}\tilde{\mu}_\Omega(t)$. Also,

$\lambda_\Omega(x_0 * (y_0 * z_0)) = \inf_{t\in f^{-1}(x'*(y'*z'))}\lambda_\Omega(t)$,

$\lambda_\Omega(y_0) = \inf_{t\in f^{-1}(y')}\lambda_\Omega(t)$

and

$\lambda_\Omega(x_0 * z_0) = \lambda_\beta\{f(x_0 * z_0)\}$

$= \lambda_\beta\{f(x'*z')\}$

$= \inf_{(x_0*z_0)\in f^{-1}(x'*z')}\lambda_\Omega(x_0 * z_0)$

$= \inf_{t\in f^{-1}(x'*z')}\lambda_\Omega(t)$. Then

$\tilde{\mu}_\beta(x'*z')$ = $r\sup_{t\in f^{-1}(x'*z')}\tilde{\mu}_\Omega(t)$ = $\tilde{\mu}_\Omega(x_0 * z_0)$

≽rmin { $\tilde{\mu}_\Omega$ (x_0∗(y_0∗z_0)), $\tilde{\mu}_\Omega$ (y_0)},

= rmin{ $r\sup_{t\in f^{-1}(x'*(y'*z'))}\tilde{\mu}_\Omega(t)$, $r\sup_{t\in f^{-1}(y')}\tilde{\mu}_\Omega(t)$ }

= rmin { $\tilde{\mu}_\beta(x'*(y'*z'))$, $\tilde{\mu}_\beta(y')$ }and

$\lambda_\beta(x'*z') = \inf_{t\in f^{-1}(x'*z')}\lambda\Omega_\Omega(t)$

$= \lambda_\Omega(x'*z')$

≤ max { $\lambda_\Omega(z_0 * (y_0 * z_0))$, $\lambda_\Omega(y_0)$ }

= max { $\inf_{t\in f^{-1}(x'*(y'*z'))}\lambda_\Omega(t)$, $\inf_{t\in f^{-1}(y')}\lambda_\Omega(t)$ }

Hence, β is a cubic AT-ideal of Y. ⌂

6. Cartesain product of cubic AT-ideals

In the section, we will provide some definition on Cartesain product of cubic AT-ideals in AT-algebras.

Definition 6.1[10]. Let Ω_1=⟨$\tilde{\mu}_{\Omega1}$,$\lambda_{\Omega1}$⟩andΩ_2=⟨$\tilde{\mu}_{\Omega2}$,$\lambda_{\Omega2}$⟩ be two cubic subsets of AT-algebras X_1 and X_2respectively.Cartesian product of cubic subsets $\Omega_1$andΩ_2 is denoted by$\Omega_1 \times \Omega_2$=⟨$\tilde{\mu}_{\Omega1\times\Omega2}$,$\lambda_{\Omega1\times\Omega2}$⟩ and is defined as, for all (x,y)∈X_1×X_2:

$\tilde{\mu}_{\Omega1\times\Omega2}$(x,y) = rmin {$\tilde{\mu}_{\Omega1}$(x),$\tilde{\mu}_{\Omega2}(y)$},$\lambda_{\Omega1\times\Omega2}$(x,y) = max {$\lambda_{\Omega1}$(x),$\lambda_{\Omega2}$(y)}.

Remark 6.2. Let X and Y be AT-algebras. We defined ∗ on X×Y by(x,y)∗(u,v) = (x∗u, y∗v) for every (x,y),(u,v)∈X×Y. Clearly (X×Y, ∗, (0,0)) is an AT-algebra.

Definition 6.3. A cubic subset$\Omega_1 \times \Omega_2$= ⟨$\tilde{\mu}_{\Omega1\times\Omega2}$,$\lambda_{\Omega1\times\Omega2}$⟩ of X_1×$X_2$is called a cubic AT-ideal of X_1×X_2 if, for all (x_1,y_1),(x_2,y_2),(x_3,y_3)∈X_1×X_2:

(1)$\tilde{\mu}_{\Omega1\times\Omega2}$(0,0)≽ $\tilde{\mu}_{\Omega1\times\Omega2}$(x,y) and$\lambda_{\Omega1\times\Omega2}$(0,0)≤ $\lambda_{\Omega1\times\Omega2}$(x,y)

(2)$\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3))\succcurlyeq$ rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$,and
$\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3))\leq\max\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))),\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$.

Theorem 6.4. Let$\Omega_1=\langle\tilde{\mu}_{\Omega 1},\lambda_{\Omega 1}\rangle$and$\Omega_2=\langle\tilde{\mu}_{\Omega 2},\lambda_{\Omega 2}\rangle$be twocubic AT-ideals of AT-algebras X_1andX_2,respectively. Then$\Omega_1\times\Omega_2=\langle\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\rangle$ is acubic AT-ideal of AT-algebra $X_1\times X_2$.

Proof. For any (x,y)∈ $X_1\times X_2$,
$\tilde{\mu}_{\Omega 1\times\Omega 2}(0,0)=$ rmin $\{\tilde{\mu}_{\Omega 1}(0),\tilde{\mu}_{\Omega 2}(0)\}\succcurlyeq$ rmin$\{\tilde{\mu}_{\Omega 1}(x),\tilde{\mu}_{\Omega 2}(y)\}=\tilde{\mu}_{\Omega 1\times\Omega 2}(x,y)$
$\lambda_{\Omega 1\times\Omega 2}(0,0)=\max\{\lambda_{\Omega 1}(0),\lambda_{\Omega 2}(0)\}\leq\max\{\lambda_{\Omega 1}(x),\lambda_{\Omega 2}(y)\}=\lambda_{\Omega 1\times\Omega 2}(x,y)$
For any$(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X_1\times X_2$,
$\tilde{\mu}_{\Omega\times\Omega 2}(x_1*x_3,y_1*y_3)=$rmin$\{\tilde{\mu}_{\Omega 1}(x_1*x_3),\tilde{\mu}_{\Omega 2}(y_1*y_3)\}$,
$\succcurlyeq$rmin{rmin$\{\tilde{\mu}_{\Omega 1}(x_1*(x_2*x_3)),\tilde{\mu}_{\Omega 1}(x_2)\}$,rmin$\{\tilde{\mu}_{\Omega 2}(y_1*(y_2*y_3)),\tilde{\mu}_{\Omega 2}(y_2)\}\}$
=rmin{rmin$\{\tilde{\mu}_{\Omega 1}(x_1*(x_2*x_3)),\tilde{\mu}_{\Omega 2}(y_1*(y_2*y_3))\}$, rmin$\{\tilde{\mu}_{\Omega 2}(x_2),\tilde{\mu}_{\Omega 2}(y_2)\}$
=rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1*(x_2*x_3)),(y_1*(y_2*y_3))),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
$\succcurlyeq$rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)*((x_2,y_2)*(x_3,y_3)),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
$\lambda_{\Omega 1\times\Omega 2}(x_1*x_3,y_1*y_3)=\max\{\lambda_{\Omega 1}(x_1*x_3),\lambda_{\Omega 2}(y_1*y_3)\}$
$\leq\max\{\max\{\lambda_{\Omega 1}(x_1*(x_2*x_3)),\lambda_{\Omega 1}(x_2)\},\max\{\lambda_{\Omega 2}(y_1*(y_2*y_3)),\lambda_{\Omega 2}(y_2)\}\}$
$=\max\{\max\{\lambda_{\Omega 1}(x_1*(x_2*x_3)),\lambda_{\Omega 2}(y_1*(y_2*y_3)),\max\{\lambda_{\Omega 1}(x_2),\lambda_{\Omega 2}(y_2)\}\}$
$=\max\{\lambda_{\Omega 1\times\Omega 2}((x_1*(x_2*x_3)),(y_1*(y_2*y_3)),\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
$\leq\max\{\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)*((x_2,y_2)*(x_3,y_3)),\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
Hence,$\Omega_1\times\Omega_2=\langle\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\rangle$ is cubic AT-ideal of AT-algebra $X_1\times X_2$. ⌂

Theorem 6.5. If$\Omega_1\times\Omega_2=\langle\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\rangle$is a cubic AT-ideal of AT- algebra $X_1\times X_2$ and if $(x_1,y_1)\leq(x_2,y_2)$, we have $\langle\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)\preccurlyeq\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\rangle$and $\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\geq\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)$,for all$(x_1,y_1),(x_2,y_2)\in X_1\times X_2$.

Proof: Let$(x_1,y_1),(x_2,y_2)\in X_1\times X_2$such that $(x_1,y_1)\leq(x_2,y_2)\Rightarrow(x_2,y_2)*(x_1,y_1)=(0,0)$.This together with
$(0,0)*(x_1,y_1)=(x_1,y_1)$=and $\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\preccurlyeq\tilde{\mu}_{\Omega 1\times\Omega 2}(0,0)$. Also, $\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\geq\lambda_{\Omega 1\times\Omega 2}(0,0)$. Consider
$\tilde{\mu}_{\Omega 1\times\Omega 2}((0,0)*(x_1,y_1))=\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)$
$\succcurlyeq\{\tilde{\mu}_{\Omega 1\times\Omega 2}((0,0)*((x_2,y_2)*(x_1,y_1))),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
= rmin $\{\tilde{\mu}_{\Omega 1\times\Omega 2}((0,0)*(0,0)),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
= rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}(0,0),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
$=\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)$
$\lambda_{\Omega 1\times\Omega 2}((0,0),(x_1,y_1))=\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)$
$\leq\{\lambda_{\Omega 1\times\Omega 2}((0,0)*(x_2,y_2)*(x_1,y_1))),\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
$=\max\{\lambda_{\Omega 1\times\Omega 2}((0,0)*(0,0)),\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
$=\max\{\lambda_{\Omega 1\times\Omega 2}(0,0),\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
$=\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)$
This shows that $\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\preccurlyeq\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)$ and $\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\geq\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)$, for all $(x_1,y_1),(x_2,y_2)\in X_1\times X_2$. ⌂

Theorem 6.6. If$\Omega_1\times\Omega_2=\langle\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\rangle$is a cubic AT-ideal of AT- algebra $X_1\times X_2$.
If$(x_1,y_1)*(x_2,y_2)\leq(x_3,y_3)$holds$X_1\times X_2$, then we have $\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\succcurlyeq$rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_3,y_3)\}$ and
$\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\leq\max\{\lambda_{\Omega 1\times\Omega 2}(x_1,y_1),\lambda_{\Omega 1\times\Omega 2}(x_3,y_3)\}$, for all $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X_1\times X_2$.

Proof Let $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X_1\times X_2$ and let$(x_1,y_1)*(x_2,y_2)\leq(x_3,y_3)$ holds in $X_1\times X_2$, then$(x_3,y_3)*((x_1,y_1)*(x_2,y_2))=(0,0)$.
Now for any (0,0)= (x_3,y_3) and from (2)

$\tilde{\mu}_{\Omega 1\times\Omega 2}((x_3,y_3)*(x_2,y_2))\succcurlyeq$rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_3,y_3)*((x_1,y_1)*(x_2,y_2))),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)\}$,
$\tilde{\mu}_{\Omega 1\times\Omega 2}((x_3,y_3)*(x_2,y_2))\tilde{\mu}_{\Omega 1\times\Omega 2}((0,0)*(x_2,y_2))=\tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)$
$\succcurlyeq$ rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_2,y_2)),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
$\succcurlyeq$rmin{rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_3,y_3)*(x_2,y_2))),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_3,y_3)\},\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
=rmin{rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_3,y_3)*((x_2,y_2)*(x_1,y_1))),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_3,y_3)\},\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
=rmin{rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}((0,0),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_3,y_3)\},\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
= rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}(x_3,y_3)\},\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
= rmin$\{\tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1),\tilde{\mu}_{\Omega 1\times\Omega 2}(x_3,y_3)\}$ and from(2)
$\lambda_{\Omega 1\times\Omega 2}((x_3,y_3)*(x_2,y_2))\leq\max\{\lambda_{\Omega 1\times\Omega 2}((x_3,y_3)*((x_1,y_1)*(x_2,y_2))),\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)\}$,
We have, $\lambda_{\Omega 1\times\Omega 2}((0,0)*(x_2,y_2))=\lambda_{\Omega 1\times\Omega 2}(x_2,y_2)$
$\leq\max\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_2,y_2)),\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
$\leq\max\{\max\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))),\lambda_{\Omega 1\times\Omega 2}(x_3,y_3)\},\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
$=\max\{\max\{\lambda_{\Omega 1\times\Omega 2}((x_3,y_3)*((x_2,y_2)*(x_1,y_1))),\lambda_{\Omega 1\times\Omega 2}(x_3,y_3)\},\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
$=\max\{\max\{\lambda_{\Omega 1\times\Omega 2}((0,0),\lambda_{\Omega 1\times\Omega 2}(x_3,y_3)\},\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
$=\max\{\lambda_{\Omega 1\times\Omega 2}(x_3,y_3),\lambda_{\Omega 1\times\Omega 2}(x_1,y_1)\}$
$=\max\{\lambda_{\Omega 1\times\Omega 2}(x_1,y_1),\lambda_{\Omega 1\times\Omega 2}(x_3,y_3)\}$.This completes the proof. ⌂

Definition 6.7. Let$\Omega_1\times\Omega_2=\{\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\}$ is a cubic AT-ideal of AT-algebra $X_1\times X_2$and for any $\tilde{t}\in D[0,1]$ and s∈ [0,1] the set
$U(\Omega_1\times\Omega_2;\tilde{t},s)=\{(x,y)\in X_1\times X_2:\tilde{\mu}_{\Omega 1\times\Omega 2}(x,y)\succcurlyeq\tilde{t},\lambda_{\Omega 1\times\Omega 2}(x,y)\leq s\}$, is called the cubic level set of$\Omega_1\times\Omega_2=\langle\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\rangle$.

Theorem 6.8. Let$\Omega_1\times\Omega_2=\{\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\}$ is a cubic subset of AT-algebra $X_1\times X_2$, then $\Omega_1\times\Omega_2=\{\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\}$ is a cubic AT-ideal of AT-algebra $X_1\times X_2$ if and only if, for any $\tilde{t}\in D[0,1]$ ands∈ [0,1] the set $U(\Omega_1\times\Omega_2;\tilde{t},s)$ is either empty or a AT-ideal of $X_1\times X_2$.

Proof: Let$\Omega_1\times\Omega_2=\{\tilde{\mu}_{\Omega 1\times\Omega 2},\lambda_{\Omega 1\times\Omega 2}\}$ is a cubic AT-ideal of AT- algebra $X_1\times X_2$, for any $\tilde{t}\in D[0,1]$ and s∈ [0,1]define the set
$U(\Omega_1\times\Omega_2;\tilde{t},s)=\{(x,y)\in X_1\times X_2:\tilde{\mu}_{\Omega 1\times\Omega 2}(x,y)\succcurlyeq\tilde{t},\lambda_{\Omega 1\times\Omega 2}(x,y)\leq s\}$.Since
$U(\Omega_1\times\Omega_2;\tilde{t},s)\neq\emptyset$, let (x,y) ∈ $U(\Omega_1\times\Omega_2;\tilde{t},s)$ implies $\tilde{\mu}_{\Omega 1\times\Omega 2}(x,y)\succcurlyeq\tilde{t}$ and$\lambda_{\Omega 1\times\Omega 2}(x,y)\leq s$. So $\tilde{\mu}_{\Omega 1\times\Omega 2}(0,0)\succcurlyeq\tilde{\mu}_{\Omega 1\times\Omega 2}(x,y)\succcurlyeq\tilde{t}$
$\Longrightarrow\tilde{\mu}_{\Omega 1\times\Omega 2}(0,0)\succcurlyeq\tilde{t},\lambda_{\Omega 1\times\Omega 2}(0,0)\leq\lambda_{\Omega 1\times\Omega 2}(x,y)\leq s\Longrightarrow\lambda_{\Omega 1\times\Omega 2}(0,0)\leq s$,This shows that (0,0) ∈$U(\Omega_1\times\Omega_2;\tilde{t},s)$.

Let $(x_1,y_1)*((x_2,y_2)*(x_3,y_3)) \in U(\Omega_1 \times \Omega_2; \tilde{t}, s)$ and $(x_2,y_2) \in U(\Omega_1 \times \Omega_2; \tilde{t}, s)$, this implies
$\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))) \succcurlyeq \tilde{t}, \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2) \succcurlyeq \tilde{t}$,
$\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))) \leq s, \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2) \leq s$.
$\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) \succcurlyeq rmin\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\} \succcurlyeq rmin\{\tilde{t}, \tilde{t}\} = \tilde{t}$
$\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) \leq max \lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$
$\leq max\{s,s\} = s$

This implied that $(x_1,y_1)*(x_3,y_3) \in U(\Omega_1 \times \Omega_2; \tilde{t}, s)$. hence, $U(\Omega_1 \times \Omega_2; \tilde{t}, s)$ is an AT-ideal of $X_1 \times X_2$.
Conversely, suppose $U(\Omega_1 \times \Omega_2; \tilde{t}, s)$ is an AT-ideal of $X_1 \times X_2$, for any $\tilde{t} \in D[0,1]$
And $s \in [0,1]$. Assume $(x_1,y_1) \in X_1 \times X_2$, such that
$\tilde{\mu}_{\Omega 1\times\Omega 2}(0,0) \prec \tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1), \lambda_{\Omega 1\times\Omega 2}(0,0) > \lambda_{\Omega 1\times\Omega 2}(x_1,y_1)$.
Put $\tilde{t}_\circ = \frac{1}{2}\{\tilde{\mu}_{\Omega 1\times\Omega 2}(0,0) + \tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)\} \Longrightarrow \tilde{\mu}_{\Omega 1\times\Omega 2}(0,0) \prec \tilde{t}_\circ \prec \tilde{\mu}_{\Omega 1\times\Omega 2}(x_1,y_1)$,
$s_\circ = \frac{1}{2}\{\lambda_{\Omega 1\times\Omega 2}(0,0) + \lambda_{\Omega 1\times\Omega 2}(x_1,y_1)\} \Longrightarrow \lambda_{\Omega 1\times\Omega 2}(0,0) > t_\circ > \lambda_{\Omega 1\times\Omega 2}(x_1,y_1)$. This implies $(x_1,y_1) \in U(\Omega_1 \times \Omega_2; \tilde{t}, s)$ but $(0,0) \notin U(\Omega_1 \times \Omega_2; \tilde{t}, s)$, which is contradiction.

Therefore $\tilde{\mu}_{\Omega 1\times\Omega 2}(0,0) \succcurlyeq \tilde{\mu}_{\Omega 1\times\Omega 2}(x,y)$ and $\lambda_{\Omega 1\times\Omega 2}(0,0) \leq \lambda_{\Omega 1\times\Omega 2}(x,y)$, for all $(x,y)) \in X_1 \times X_2$. Assum $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in X_1 \times X_2$ such that
$\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) < rmin\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$.
Let $\tilde{t}_\circ = \frac{1}{2}\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) + rmin\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}\}$
Then $\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) \prec \tilde{t}_\circ \prec rmin\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$. Also
$\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) > max\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$.
Let $s_\circ = \frac{1}{2}\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) + max\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}\}$. Then
$\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) > s_\circ > max\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$.
This show that $(x_1,y_1)*((x_2,y_2)*(x_3,y_3)) \in U(\Omega_1 \times \Omega_2; \tilde{t}, s)$, $(x_2,y_2) \in U(\Omega_1 \times \Omega_2; \tilde{t}, s)$.
But $(x_1,y_1)*(x_3,y_3) \notin U(\Omega_1 \times \Omega_2; \tilde{t}, s)$ which is a contradiction, therefore
$\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) \succcurlyeq rmin\{\tilde{\mu}_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \tilde{\mu}_{\Omega 1\times\Omega 2}(x_2,y_2)\}$.
Similarly,
$\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*(x_3,y_3)) \leq max\{\lambda_{\Omega 1\times\Omega 2}((x_1,y_1)*((x_2,y_2)*(x_3,y_3))), \lambda_{\Omega 1\times\Omega 2}(x_2,y_2)\}$.
Hence $\Omega_1 \times \Omega_2 = \{\tilde{\mu}_{\Omega 1\times\Omega 2}, \lambda_{\Omega 1\times\Omega 2}\}$ is a cubic AT-ideal of AT-algebra $X_1 \times X_2$. ⌂

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