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Elements of the Homometric Vector K-Product Part I

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Abstract
This paper aims to generalize the usual vector product between two given vectors, defined by Gibbs and Heaviside, from three-space to n-space. Next, although not very intuitive, this idea will be generalized to define an axial vector simultaneously orthogonal to any k given vectors of an n-dimensional vector space $\mathbb{H}^n$, with $2 \leq k \in \mathbb{N}$, considering the orthogonality condition defined by the usual scalar product. The vector product thus generalized is given the name homometric vector k-product, because an axial vector of $\mathbb{H}^n$, whose components are solutions of a homogeneous linear system of k equations with n unknowns, results from this product. For this first part, some specific properties of the homometric vector product will be analyzed, highlighting the high theoretical and practical value of this operation for science and engineering.

Keywords: Vector product, homometric k-product, n-dimensional vector spaces.

1. Introduction

The usual vector product is a very useful vector operation in science and engineering, participating in various concepts and operations such as rotational, surface integrals, angular momentum, changes of variables in multiple integrals, Lorentz force, orientation of surfaces, among other related. Thus, one of the major limitations of this operation is the fact that its validity occurs only in three-dimensional vector spaces, which imposes the use of tensor calculus, often with limiting and not very enlightening clippings about the nature of the phenomena studied.

This paper aims to generalize this usual vector product, making it valid for vector spaces of any dimensions. Thus, as certain properties of this product valid in three-dimensional spaces, such as the Jacobi identity, are not valid in all dimensions, and taking into account the need to introduce a more general definition to characterize this product of vectors, there is an urgent need to call the generalized vector product a homometric vector k-product.

The homometric vector k-product of n-dimensional vector space is a vector operation from which results an axial vector simultaneously orthogonal to the given k vectors. It generalizes the classical vector product, valid only on $\mathbb{R}^3$ and for 2 multiplicative vectors, to n-dimensional vector spaces and product of k vectors (Anton & Rorres, 2012).

Hence, the classical vector product, denoted by $\boldsymbol{v}_0 \times \boldsymbol{v}_1$, is an antisymmetric bilinear operation from which also results a vector simultaneously orthogonal to the two given vectors, whose modulus is $\|\boldsymbol{v}_0 \times \boldsymbol{v}_1\| = \|\boldsymbol{v}_0\|\|\boldsymbol{v}_1\| \sin\theta$, where θ is the angle between these vectors (Hoffman & Kunze, 1971), and also admits an algebraic form (EVES, 2011). Thus, this vector product is a particular case of the homometric vector 2-product.

2. Materials and methods

For the achievement of the objectives of this work, exploratory research was used, with logical-deductive methods (HOEFEL, 2002), aiming to define the homometric vector k-product and present some properties and functionalities of the homometric vector 2-product. For the calculation of components of homometric vector k-product, the systematic dimensional shrinkage method was used.

3. Definition of the Homometric Vector K-Product

The definition of the homometric vector k-product presented in this article is quite general, and the different specific products, classified according to the number of vectors multiplied, should be treated in parts, being this article focused to the homometric vector 2-product of only two multiplicative vectors.

Definition 01. Be $\mathbb{H}^n$ a vector space over $\mathbb{R}$, and $\boldsymbol{v}_1 = (a_{11}; \dots; a_{1n-1}; a_{1n})$, $\boldsymbol{v}_2 = (a_{21}; \dots; a_{2n-1}; a_{2n})$ $(a_{21}; \dots; a_{2n-1}; a_{2n}); \dots;$ $\boldsymbol{v}_k = (a_{31}; \dots; a_{3n-1}; a_{3n}) \in \mathbb{H}^n$, k vectors. A multilinear application, $\circledast: \mathbb{H}_1^n \times \dots \times \mathbb{H}_k^n \to \mathbb{H}^n$, is called homometric vector k-product if:

i) The axial vector $\boldsymbol{v}_1 \circledast \boldsymbol{v}_2 \circledast \dots \circledast \boldsymbol{v}_k$ is simultaneously orthogonal to the k vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$, given;

ii) $\boldsymbol{v}_1 \circledast \boldsymbol{v}_2 \circledast \dots \circledast \boldsymbol{v}_k = (\Delta x_1; \Delta x_2; \Delta x_3; \dots; \Delta x_{n-1}; \Delta) \in \mathbb{H}^n$, if and only if there is a homogeneous linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_{13} + \dots + a_{1n-1}x_{n-1} + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n-1}x_{n-1} + a_{2n}x_n = 0 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \dots + a_{kn-1}x_{n-1} + a_{kn}x_n = 0 \end{cases} \quad (1)$$

which can be converted into, at least, one Cramer system, such that the following equalities are fulfilled:

$$\frac{x_1}{x_n} = \frac{\Delta x_1}{\Delta}; \frac{x_2}{x_n} = \frac{\Delta x_2}{\Delta}; \dots; \frac{x_{n-1}}{x_n} = \frac{\Delta x_{n-1}}{\Delta}; \ x_n = \Delta, \forall \Delta \neq 0; \ x_1, x_2, \dots, x_{n-1}, x_n \in \mathbb{R}.$$

As already referenced above, for this first part of the results of this research, only the homometric vector 2-product, for $k = 2$, of $\mathbb{H}^n$, will be analyzed.

Proposition 01. Be $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k \in \mathbb{H}^n$. The components $\Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_{n-1}, \Delta$ of the axial vector $\boldsymbol{v}_1 \circledast \boldsymbol{v}_2 \circledast \dots \circledast \boldsymbol{v}_k$ are solutions of the homogeneous linear system (01).

Demonstration. Considering the homogeneous linear system (01) and dividing the k equations by $x_n \neq 0$, we have below.

$$\begin{cases} a_{11}\frac{x_1}{x_n} + a_{12}\frac{x_2}{x_n} + a_{13}\frac{x_3}{x_n} + \dots + a_{1n-1}\frac{x_{n-1}}{x_n} = -a_{1n} \\ a_{21}\frac{x_1}{x_n} + a_{22}\frac{x_2}{x_n} + a_{23}\frac{x_3}{x_n} + \dots + a_{2n-1}\frac{x_{n-1}}{x_n} = -a_{2n} \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{k1}\frac{x_1}{x_n} + a_{k2}\frac{x_2}{x_n} + a_{k3}\frac{x_3}{x_n} + \dots + a_{kn-1}\frac{x_{n-1}}{x_n} = -a_{kn} \end{cases}$$

Given definition 01, it follows:

$$\begin{cases} a_{11}\frac{\Delta x_1}{\Delta} + a_{12}\frac{\Delta x_2}{\Delta} + a_{13}\frac{\Delta x_3}{\Delta} + \dots + a_{1n-1}\frac{\Delta x_{n-1}}{\Delta} = -a_{1n} \\ a_{21}\frac{\Delta x_1}{\Delta} + a_{22}\frac{\Delta x_2}{\Delta} + a_{23}\frac{\Delta x_3}{\Delta} + \dots + a_{2n-1}\frac{\Delta x_{n-1}}{\Delta} = -a_{2n} \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{k1}\frac{\Delta x_1}{\Delta} + a_{k2}\frac{\Delta x_2}{\Delta} + a_{k3}\frac{\Delta x_3}{\Delta} + \dots + a_{kn-1}\frac{\Delta x_{n-1}}{\Delta} = -a_{kn} \end{cases}.$$

From this last linear system we conclude, as we wanted to demonstrate, that $\Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_{n-1}, \Delta$ are solutions of the homogeneous linear system (01):

$$\begin{cases} a_{11}\Delta x_1 + a_{12}\Delta x_2 + a_{13}\Delta x_3 + \dots + a_{1n-1}\Delta x_{n-1} + a_{1n}\Delta = 0 \\ a_{21}\Delta x_1 + a_{22}\Delta x_2 + a_{23}\Delta x_3 + \dots + a_{2n-1}\Delta x_{n-1} + a_{2n}\Delta = 0 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{k1}\Delta x_1 + a_{k2}\Delta x_2 + a_{k3}\Delta x_3 + \dots + a_{kn-1}\Delta x_{n-1} + a_{kn}\Delta = 0 \end{cases}.$$

Proposition 02. Be $\boldsymbol{v}_0, \boldsymbol{v}_1 \in \mathbb{H}^n$. The components of $\boldsymbol{v}_0 \circledast \boldsymbol{v}_1$ are uniquely calculated by systematically reducing the number of unknowns of the associated homogeneous system from n to $n = 3$.

Demonstration. Consider the following homogeneous linear system that defines the homometric vector 2-product:

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_{n-1}x_{n-1} + a_nx_n = 0 \\ b_1x_1 + b_2x_2 + b_3x_3 + \dots + b_{n-1}x_{n-1} + b_nx_n = 0 \end{cases}$$

Cramer's theorem requires that the given system have the number of equations equal to the number of unknowns. Thus, the system of definition 01 can only be Cramer's if:

i) The number of unknowns reduces to $n = 2$;

ii) Transform this system of definition 01 into a non-homogeneous linear system.

To transform this homogeneous linear system into Cramer's system, must first reduce the number of unknowns to $n = 3$ and then, using the following procedure, transform it from homogeneous to non-homogeneous:

$$\begin{cases} a_1x_1 + a_2x_2 = -a_3x_3 / \div x_3 \neq 0 \\ b_1x_1 + b_2x_2 = -b_3x_3 / \div x_3 \neq 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1\frac{x_1}{x_3} + a_2\frac{x_2}{x_3} = -a_3 \\ b_1\frac{x_1}{x_3} + b_2\frac{x_2}{x_3} = -b_3 \end{cases}. (1)$$

So, there are numbers $\Delta, \Delta x_1, \Delta x_2, \forall \Delta \neq 0$, such that, as it was intended to demonstrate:

$$\Delta = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}; \ \frac{x_1}{x_3} = \frac{\Delta x_1}{\Delta},$$

$$\Delta x_1 = \begin{vmatrix} -a_3 & a_2 \\ -b_3 & b_2 \end{vmatrix}; \ \frac{x_2}{x_3} = \frac{\Delta x_2}{\Delta},$$

$$\Delta x_2 = \begin{vmatrix} a_1 & -a_3 \\ b_1 & -b_3 \end{vmatrix}.$$

Definition 02. The method of obtaining the components of $\boldsymbol{v}_0 \circledast \boldsymbol{v}_1$ from the homogeneous system that defines it by systematically reducing the number of its unknowns is called systematic dimensional contraction.

This method follows the steps below, considering the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ (ZAU, 2021):

Example 01. Using systematic dimensional contraction, calculate $\boldsymbol{v}_0 \circledast \boldsymbol{v}_1 \in \mathbb{R}^4$.

Step 1.

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0 \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0 \end{cases}.$$

Decompose the unknowns into sums of $n - 1$ subunits and replace them in this system.

$$n = 4 \Rightarrow \begin{cases} x_1 = x_{11} + x_{12} + x_{13} \\ x_2 = x_{21} + x_{22} + x_{23} \\ x_3 = x_{31} + x_{32} + x_{33} \\ x_4 = x_{41} + x_{42} + x_{43} \end{cases}.$$

$$\Leftrightarrow \begin{cases} a_1(x_{11} + x_{12} + x_{13}) + a_2(x_{21} + x_{22} + x_{23}) + \\ \quad + a_3(x_{31} + x_{32} + x_{33}) + \\ \quad + a_4(x_{41} + x_{42} + x_{43}) = 0 \\ b_1(x_{11} + x_{12} + x_{13}) + b_2(x_{21} + x_{22} + x_{23}) + \\ \quad + b_3(x_{31} + x_{32} + x_{33}) + \\ \quad + b_4(x_{41} + x_{42} + x_{43}) = 0 \end{cases}.$$

Step 2. Form a system from each homogeneous equation in Step 1:

$$\begin{cases} a_1x_{11} + a_2x_{21} + a_3x_{31} + 0 = 0 \\ a_1x_{12} + a_2x_{22} + 0 + a_4x_{41} = 0 \\ a_1x_{13} + 0 + a_3x_{32} + a_4x_{42} = 0 \\ 0 + a_2x_{23} + a_3x_{33} + a_4x_{43} = 0 \end{cases}$$

$$a_1(x_{11} + x_{12} + x_{13}) + a_2(x_{21} + x_{22} + x_{23}) + a_3(x_{31} + x_{32} + x_{33}) + a_4(x_{41} + x_{42} + x_{43}) = 0$$

$$\begin{cases} b_1x_{11} + b_2x_{21} + b_3x_{31} + 0 = 0 \\ b_1x_{12} + b_2x_{22} + 0 + b_4x_{41} = 0 \\ b_1x_{13} + 0 + b_3x_{32} + b_4x_{42} = 0 \\ 0 + b_2x_{23} + b_3x_{33} + b_4x_{43} = 0 \end{cases}$$

$$b_1(x_{11} + x_{12} + x_{13}) + b_2(x_{21} + x_{22} + x_{23}) + b_3(x_{31} + x_{32} + x_{33}) + b_4(x_{41} + x_{42} + x_{43}) = 0$$

Step 3. Form the partial systems of two equations extracted from each of the two systems formed above, allowing to extract the solutions Δx_{kj}.

$$1) \begin{cases} a_1x_{11} + a_2x_{21} + a_3x_{31} = 0 \\ \\ b_1x_{11} + b_2x_{21} + b_3x_{31} = 0 \end{cases} \Rightarrow \begin{cases} \Delta x_{11} = a_2b_3 - b_2a_3 \\ \Delta x_{21} = a_3b_1 - a_1b_3 \\ \Delta x_{31} = a_1b_2 - a_2b_1 \end{cases}$$

$$2) \begin{cases} a_1x_{12} + a_2x_{22} + a_4x_{41} = 0 \\ \\ b_1x_{12} + b_2x_{22} + b_4x_{41} = 0 \end{cases} \Rightarrow \begin{cases} \Delta x_{12} = a_2b_4 - a_4b_2 \\ \Delta x_{22} = a_4b_1 - a_1b_4 \\ \Delta x_{41} = a_1b_2 - b_1a_2 \end{cases}$$

$$3) \begin{cases} a_1x_{13} + a_3x_{32} + a_4x_{42} = 0 \\ \\ b_1x_{13} + b_3x_{32} + b_4x_{42} = 0 \end{cases} \Rightarrow \begin{cases} \Delta x_{13} = a_3b_1 - a_4b_3 \\ \Delta x_{32} = a_4b_1 - a_1b_4 \\ \Delta x_{42} = a_1b_3 - a_3b_1 \end{cases}$$

$$4) \begin{cases} a_2x_{23} + a_3x_{33} + a_4x_{43} = 0 \\ \\ b_2x_{23} + b_3x_{33} + b_4x_{43} = 0 \end{cases} \Rightarrow \begin{cases} \Delta x_{23} = a_3b_4 - a_4b_3 \\ \Delta x_{33} = a_4b_2 - a_2b_4 \\ \Delta x_{43} = a_2b_3 - a_3b_2 \end{cases}$$

.

$$\Rightarrow \begin{cases} \Delta x_1 = a_2b_3 - b_2a_3 + a_2b_4 - a_4b_2 + a_3b_4 - a_4b_3 \\ \Delta x_2 = a_3b_1 - a_1b_3 + a_4b_1 - a_1b_4 + a_3b_4 - a_4b_3 \\ \Delta x_3 = a_1b_2 - a_2b_1 + a_4b_1 - a_1b_4 + a_4b_2 - a_2b_4 \\ \Delta x_4 = a_1b_2 - b_1a_2 + a_1b_3 - a_3b_1 + a_2b_3 - a_3b_2 \end{cases}.$$

4. Some Properties of the Homometric Vector 2-Product on $\mathbb{R}^n$

Theorem 01. Consider $\boldsymbol{v}_0 = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + \cdots + a_n\boldsymbol{e}_n$ and $\boldsymbol{v}_1 = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 + \cdots + b_n\boldsymbol{e}_n$. The axial vector $\boldsymbol{v}_0 \otimes \boldsymbol{v}_1$ can be calculated by multiplying these vectors:

$$(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + \cdots + a_n\boldsymbol{e}_n) \otimes (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 + \cdots + b_n\boldsymbol{e}_n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(\begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_j \right)$$

Demonstration. Considering these vectors: $\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = (a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + \cdots + a_n\boldsymbol{e}_n) \otimes (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 + \cdots + b_n\boldsymbol{e}_n)$.

Multiplying this right member, follows:

$$\begin{aligned} \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = {} & a_1b_1\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + a_1b_2\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + a_1b_3\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + \cdots + a_1b_n\boldsymbol{e}_1 \otimes \boldsymbol{e}_n + a_2b_1\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \\ & + a_2b_2\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + a_2b_3\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + \cdots + a_2b_n\boldsymbol{e}_2 \otimes \boldsymbol{e}_n + a_3b_1\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + a_3b_2\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + a_3b_3\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \\ & + \cdots + a_3b_n\boldsymbol{e}_3 \otimes \boldsymbol{e}_n + \cdots + a_nb_1\boldsymbol{e}_n \otimes \boldsymbol{e}_1 + a_nb_2\boldsymbol{e}_n \otimes \boldsymbol{e}_2 + + a_nb_3\boldsymbol{e}_n \otimes \boldsymbol{e}_3 + \cdots + a_nb_n\boldsymbol{e}_n \otimes \boldsymbol{e}_n. \end{aligned}$$

Given corollary 01, we have $\boldsymbol{e}_k \otimes \boldsymbol{e}_k = 0$:

$$\begin{aligned} \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = {} & a_1b_2\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + a_1b_3\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + \cdots + a_1b_n\boldsymbol{e}_1 \otimes \boldsymbol{e}_n + a_2b_1\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + a_2b_3\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ & + \cdots + a_2b_n\boldsymbol{e}_2 \otimes \boldsymbol{e}_n + a_3b_1\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + a_3b_2\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + \cdots + a_3b_n\boldsymbol{e}_3 \otimes \boldsymbol{e}_n + \cdots + a_nb_1\boldsymbol{e}_n \otimes \boldsymbol{e}_1 \\ & + a_nb_2\boldsymbol{e}_n \otimes \boldsymbol{e}_2 + a_nb_3\boldsymbol{e}_n \otimes \boldsymbol{e}_3 + \cdots + a_nb_{n-1}\boldsymbol{e}_n \otimes \boldsymbol{e}_{n-1}. \end{aligned}$$

Considering definition 01:

$$\begin{aligned} \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = {} & (a_2b_1 - a_1b_2)\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + (a_3b_1 - a_1b_3)\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + \cdots + (a_nb_1 - a_1b_n)\boldsymbol{e}_n \otimes \boldsymbol{e}_1 + (a_3b_2 - a_2b_3)\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + \cdots \\ & + (a_nb_2 - a_2b_n)\boldsymbol{e}_n \otimes \boldsymbol{e}_2 + \cdots + (a_nb_3 - a_3b_n)\boldsymbol{e}_n \otimes \boldsymbol{e}_3 + + \cdots + (a_nb_{n-1} - a_{n-1}b_i)\boldsymbol{e}_n \otimes \boldsymbol{e}_{n-1}. \end{aligned}$$

Using the summation symbol:

$$\begin{aligned} \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = {} & \sum_{k=2}^{n} (a_kb_1 - a_1b_k)\boldsymbol{e}_k \otimes \boldsymbol{e}_1 + \\ & + \sum_{k=3}^{n} (a_kb_2 - a_2b_k)\boldsymbol{e}_k \otimes \boldsymbol{e}_2 + \sum_{k=4}^{n} (a_kb_3 - a_3b_k)\boldsymbol{e}_k \otimes \boldsymbol{e}_3 + \cdots \\ & + \sum_{k=n}^{n} (a_kb_{n-1} - a_{n-1}b_k)\boldsymbol{e}_k \otimes \boldsymbol{e}_n. \end{aligned}$$

Considering $a_kb_j - a_jb_k = \begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix}$, we have:

$$\begin{aligned} \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = {} & \sum_{k=2}^{n} \begin{vmatrix} a_k & a_1 \\ b_k & b_1 \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_1 + \sum_{k=3}^{n} \begin{vmatrix} a_k & a_2 \\ b_k & b_2 \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_2 \\ & + \sum_{k=n}^{n} \begin{vmatrix} a_k & a_3 \\ b_k & b_3 \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_3 + \cdots \\ & + \sum_{k=n}^{n} \begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_{n-1}. \end{aligned}$$

The right member contains a sum (from 1 to $n-1$ plots) of other sums (from $j+1$ to n plots), whence, finally:

$$\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(\begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_j \right) = $$

$$= (a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + \cdots + a_n\boldsymbol{e}_n) \otimes (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 + \cdots + b_n\boldsymbol{e}_n).$$

The calculation of $\boldsymbol{v}_0 \otimes \boldsymbol{v}_1$ from this expression of the theorem 01 requires the calculation of the homometric 2-product of the versors, using the definition 01.

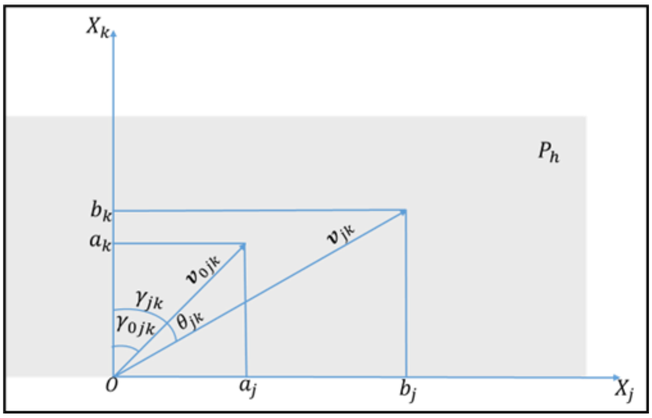
Theorem 02. Be $\boldsymbol{v}_0 \otimes \boldsymbol{v}_1$, $\boldsymbol{v}_0$, $\boldsymbol{v}_1 \in \mathbb{H}^n$. Then, there exist component vectors $\boldsymbol{v}_{0jk}$ and $\boldsymbol{v}_{jk}$ such that, for angles θ_{jk} between these vectors, the equality below is fulfilled:

$$\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(\|\boldsymbol{v}_{0jk}\| \|\boldsymbol{v}_{jk}\| \sin\theta_{jk} \, \boldsymbol{e}_k \otimes \boldsymbol{e}_j \right).$$

Demonstration. From Theorem 01 it is known that:

$$\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(\begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_j \right).$$

Considering $\begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} = a_k b_j - a_j b_k$, there exist angles γ_{0jk} and γ_{jk}, in the planes $X_j X_k$, $k \neq j \in \mathbb{N}$, of any n-dimensional coordinate system $OX_1X_2 \dots X_n$, such that $a_j = \|\boldsymbol{v}_{0jk}\| \sin \gamma_{0jk}$, $a_k = \|\boldsymbol{v}_{0jk}\| \cos \gamma_{0jk}$, $b_j = \|\boldsymbol{v}_{jk}\| \sin \gamma_{jk}$, $b_k = \|\boldsymbol{v}_{jk}\| \cos \gamma_{jk}$
are orthogonal projections of $\boldsymbol{v}_{0jk}$, $\boldsymbol{v}_{jk}$ on these planes $X_j X_k$, according to the following figure.



Source: Own elaboration.
Fig. 01: Representation of vectors $\boldsymbol{v}_{0jk}, \boldsymbol{v}_{jk}$ and angles $\gamma_{0jk}, \gamma_{jk}$.

From this figure 01, follows:

$$\begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} = \|\boldsymbol{v}_{0jk}\| \cos \gamma_{0jk} \|\boldsymbol{v}_{jk}\| \sin \gamma_{jk} - \|\boldsymbol{v}_{jk}\| \cos \gamma_{jk} \|\boldsymbol{v}_{0jk}\| \sin \gamma_{0jk}$$

$$\Rightarrow \begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} = \|\boldsymbol{v}_{0jk}\| \|\boldsymbol{v}_{jk}\| (\cos \gamma_{0jk} \sin \gamma_{jk} - \cos \gamma_{jk} \sin \gamma_{0jk}) .$$

Thus, there exist angles $\theta_{jk} = \gamma_{jk} - \gamma_{0jk}$ such that, as it was intended to show:

$$\begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} = \|\boldsymbol{v}_{0jk}\| \|\boldsymbol{v}_{jk}\| \sin \theta_{jk}$$

$$\Leftrightarrow \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(\|\boldsymbol{v}_{0jk}\| \|\boldsymbol{v}_{jk}\| \sin \theta_{jk}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_j \right) .$$

Theorem 03. Be $\boldsymbol{v}_0 \otimes \boldsymbol{v}_1$, $\boldsymbol{v}_0$, $\boldsymbol{v}_1 \in \mathbb{H}^n$ e $\|\boldsymbol{v}_0 \otimes \boldsymbol{v}_1\|$, $\|\boldsymbol{v}_0\|$, $\|\boldsymbol{v}_1\|$ their modules, respectively. Being θ the angle between the vectors $\boldsymbol{v}_0$, $\boldsymbol{v}_1 \in \mathbb{H}^n$, then follows:

$$\|\boldsymbol{v}_0 \otimes \boldsymbol{v}_1\| = \|\boldsymbol{v}_0\| \|\boldsymbol{v}_1\| \sin \theta .$$

Demonstration. It is known from Theorem 02 that:

$$\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(\|\boldsymbol{v}_{0jk}\| \|\boldsymbol{v}_{jk}\| \sin \theta_{jk}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_j \right) .$$

Considering the plane $\chi\psi$ that contains the vectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$, the calculation of the homometric vector 2-product between these two vectors, in this plane, follows below:

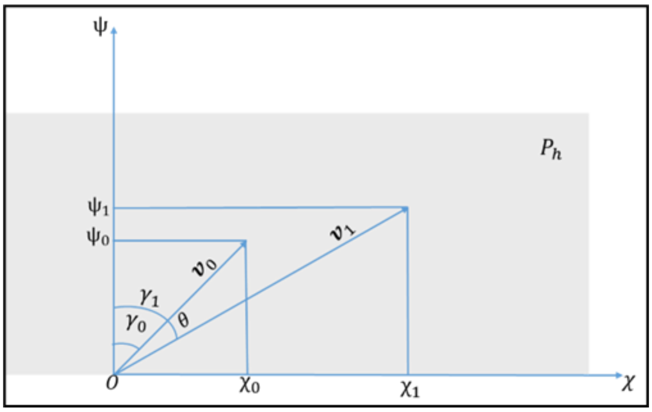
$$\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = \sum_{j=1}^{1} \sum_{k=2}^{2} \left(\|\boldsymbol{v}_{0jk}\| \|\boldsymbol{v}_{jk}\| \sin \theta_{jk}\, \boldsymbol{e}_\psi \otimes \boldsymbol{e}_\chi \right) .$$

$$\Rightarrow \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = \|\boldsymbol{v}_{012}\| \|\boldsymbol{v}_{12}\| \sin \theta_{12}\, \boldsymbol{e}_\psi \otimes \boldsymbol{e}_\chi$$

Now, in this case of the plan $\chi\psi$, $\|\boldsymbol{v}_{012}\| = \|\boldsymbol{v}_0\|$, $\|\boldsymbol{v}_{12}\| = \|\boldsymbol{v}_1\|$ and $\theta_{12} = \theta$. Then, we have, as we wanted to show:

$$\|\boldsymbol{v}_0 \otimes \boldsymbol{v}_1\| = \|\boldsymbol{v}_0\| \|\boldsymbol{v}_1\| \sin \theta .$$

The plane $\chi\psi$ or P_h, which follows represented in figure 02 below, is called homometric plane.



Source: Own elaboration.
Fig. 02: Representation of vectors v_0, v_1 and angles $\gamma_0, \gamma_1, \theta$ in the homometric plane.

Theorem 04. Let $\mathbb{H}$ be a vector space over $\mathbb{R}$. Then, the axial vector $\boldsymbol{v}_0 \otimes \boldsymbol{v}_1$ satisfies the following equality:

$$\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = -\, \boldsymbol{v}_1 \otimes \boldsymbol{v}_0 .$$

Demonstration. Considering a_i, a_j, as components of $\boldsymbol{v}_0$, and b_i, b_j, as components of $\boldsymbol{v}_1$, follows below:

$$\boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(\begin{vmatrix} a_k & a_j \\ b_k & b_j \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_j \right)$$

$$\Rightarrow \boldsymbol{v}_1 \otimes \boldsymbol{v}_0 = -\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \left(\begin{vmatrix} b_k & b_j \\ a_k & a_j \end{vmatrix} \boldsymbol{e}_k \otimes \boldsymbol{e}_j \right).$$

$$\Leftrightarrow \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 = -\, \boldsymbol{v}_1 \otimes \boldsymbol{v}_0 .$$

Corolary 01. The homometric vector 2-product of a vector with itself is null:

$$\boldsymbol{v} \otimes \boldsymbol{v} = \boldsymbol{0} .$$

Demonstration. Consider the vector $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1$. Then:

$$(\boldsymbol{v}_0 + \boldsymbol{v}_1) \otimes (\boldsymbol{v}_0 + \boldsymbol{v}_1) = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{v}_0 \otimes \boldsymbol{v}_0 + \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 + \boldsymbol{v}_1 \otimes \boldsymbol{v}_0 + \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 = 0 .$$

$$\Rightarrow \boldsymbol{v}_0 \otimes \boldsymbol{v}_0 + \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 - \boldsymbol{v}_0 \otimes \boldsymbol{v}_1 + \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 = 0$$

$$\Rightarrow \boldsymbol{v}_0 \otimes \boldsymbol{v}_0 + \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 = 0 \Leftrightarrow \boldsymbol{v}_0 \otimes \boldsymbol{v}_0 = -\boldsymbol{v}_1 \otimes \boldsymbol{v}_1 .$$

So, we have, as wanted to demonstrate:

$$\boldsymbol{v}_0 \otimes \boldsymbol{v}_0 - \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 = 0 \Leftrightarrow \boldsymbol{v}_0 \otimes \boldsymbol{v}_0 = \boldsymbol{v}_1 \otimes \boldsymbol{v}_1 = 0 .$$

Theorem 05. Consider $\boldsymbol{v}_0, \boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \mathbb{H}^n$. The homometric vector 2-product of these three vectors does not, in general, comply with Jacobi's identity:

$$\boldsymbol{v}_0 \otimes (\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) + \boldsymbol{v}_1 \otimes (\boldsymbol{v}_2 \otimes \boldsymbol{v}_0) + \boldsymbol{v}_2 \otimes (\boldsymbol{v}_0 \otimes \boldsymbol{v}_1) \neq \boldsymbol{0} \;\; \forall \boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{H}^{\boldsymbol{n}} .$$

Demonstration. A counterexample is enough to verify that Jacobi's identity does not hold in $\mathbb{H}^n$. Take, as multiplicative vectors, versors of the canonical basis $B = \langle \boldsymbol{e}_1; \boldsymbol{e}_2;\ \boldsymbol{e}_3;\ \boldsymbol{e}_4;\ \boldsymbol{e}_5 \rangle$:

$$\boldsymbol{e}_1 \otimes (\boldsymbol{e}_2 \otimes \boldsymbol{e}_3) + \boldsymbol{e}_2 \otimes (\boldsymbol{e}_3 \otimes \boldsymbol{e}_1) + \boldsymbol{e}_3 \otimes (\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) \neq \boldsymbol{0}.$$

Considering theorem 02 above, it follows:

$$\boldsymbol{e}_1 \otimes (\boldsymbol{e}_4 + \boldsymbol{e}_5 + \boldsymbol{e}_1) + \boldsymbol{e}_2 \otimes (\boldsymbol{e}_2 - \boldsymbol{e}_4 - \boldsymbol{e}_5) + \boldsymbol{e}_3 \otimes (\boldsymbol{e}_3 + \boldsymbol{e}_4 + \boldsymbol{e}_1) \neq \boldsymbol{0}.$$

$$\Rightarrow \boldsymbol{e}_1 \otimes \boldsymbol{e}_4 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_5 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_4 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_5 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_4 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_1 \neq \boldsymbol{0}.$$

Applying elementary properties of the 2-product, it follows:

$$\boldsymbol{e}_1 \otimes \boldsymbol{e}_4 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_5 + \boldsymbol{0} + \boldsymbol{0} - \boldsymbol{e}_2 \otimes \boldsymbol{e}_4 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_5 + \boldsymbol{0} + \boldsymbol{e}_3 \otimes \boldsymbol{e}_4 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_1 \neq \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{e}_1 \otimes \boldsymbol{e}_4 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_5 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_4 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_5 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_4 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_1 \neq \boldsymbol{0}.$$

Applying the definition 01 again:

$$\boldsymbol{e}_5 - \boldsymbol{e}_2 - \boldsymbol{e}_3 - \boldsymbol{e}_2 - \boldsymbol{e}_3 - \boldsymbol{e}_4 + \boldsymbol{e}_3 - \boldsymbol{e}_1 - \boldsymbol{e}_5 + \boldsymbol{e}_3 + \boldsymbol{e}_4 - \boldsymbol{e}_1 + \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_5 +$$

$+ \boldsymbol{e}_2 - \boldsymbol{e}_4 - \boldsymbol{e}_5 \neq \boldsymbol{0}.$

From this last expression, we finally have:

$$-\boldsymbol{e}_4 - \boldsymbol{e}_1 \neq 0\ .$$

Concluding Remarks

After a brief review of the definition and some properties of the homometric vector 2-product, some concluding remarks can be made, as follows below:

1. The axial vector $\boldsymbol{v}_0 \circledS \boldsymbol{v}_1$ is simultaneously orthogonal to the two given multiplicative vectors, $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$, by definition, and results from an antisymmetric bilinear operation between these two vectors given;
2. This axial $\boldsymbol{v}_0 \circledS \boldsymbol{v}_1$ is zero if one of its multiplicative vectors is zero, both of its vectors are zero, or one of its given vectors is a multiple of the other;
3. The axial vector $\boldsymbol{v}_0 \circledS \boldsymbol{v}_1$ belongs to the same vector space as the two multiplicative vectors, $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$, thus fulfilling the closure property of this vector operation;
4. 2-vector belongs to the outer product, while the expression 2-product belongs to the homometric product. The result of the outer product is a bivector or 2-vector, denoted by $\boldsymbol{v}_0 \wedge \boldsymbol{v}_1$, while the result of the homometric vector 2-product is a axial vector, denoted by $\boldsymbol{v}_0 \circledS \boldsymbol{v}_1 \in \mathbb{H}^n$;
5. The necessary condition for a second-order antisymmetric tensor to be a 2-vector is that its components satisfy the Jacobi identity, when the homometric vector 2-product does not, in general, satisfy the Jacobi identity.

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