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Mathematical Bases of the Finite Element Method for Solving Elastic Static Axisymmetric Problems.

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Abstract
Some aspects of the use and justification of the finite element method, which is most widely used at present for the machine solution of many of the most important problems of mathematical physics, are considered. The method will be formulated in its simplest version using the example of a one-dimensional problem; here, the connection with the classical Rayleigh-Ritz method will be shown. Next, some existing generalizations of the method to the case of boundary value problems in more complex domains will be shown.

Keywords: occupied by a perfectly elastic homogeneous and isotropic medium, points of the rod, longitudinal deformation, normal stress in sections

Introduction

Let there be an area Ω, limited to some surface of revolution S and two flat sections S_0 and S_1, perpendicular to the axis of rotation (Fig. 1), occupied by a perfectly elastic homogeneous and isotropic medium (elastic rod of variable cross section). Assume that the section S_0 tightly pinched, S_1 free of effort. Axis of rotation compatible with axis OX Cartesian system with a beginning in the left section. We also assume that mass forces with density $F = F(x)$, parallel to the axis of rotation. It is required to determine the displacements of all points of the considered region.

We will assume that this problem can be solved as one-dimensional, i.e. that all characteristics of the stress and strain state depend only on the coordinate X . Denote by $U = U(x)$ moving points of the bar, $\varepsilon = \dfrac{dU}{dx}$ longitudinal strain, $\sigma = \sigma(x)$ normal stress in sections perpendicular to the axis OX , $S(x)$ areas of these sections, where E – Young's modulus.

The total gain P in any cross section is

$$P = P(x) = \delta \cdot S(x) = E \cdot S(x) \cdot \frac{dU}{dx} \qquad (1)$$

The differential equilibrium equation has the form

$$\frac{d}{d(x)}\left(S(x)\cdot\frac{dU}{dx}\right) + \frac{F(x)\cdot S(x)}{E} = 0 \qquad (2)$$

The nature of the fixing of the edges of the rod allows you to determine the type of boundary conditions

$$U\big|_{X=0} = 0; \qquad \left(E\cdot S(x)\cdot\frac{dU}{dx}\right)\bigg|_{X=2} = 0 \qquad (3)$$

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Thus, the stated mechanical problem was reduced to a boundary value problem. For an approximate solution of the original problem, we apply the following technique. Mentally cut the rod into sections

$x = x_k = const;$ $x_k = \frac{L}{n} \cdot k,$ $k = 1,2,....., n$ and suppose that within each section (element) $x_{k-1} \le x \le x_k,$ $k = 1,2,...., n,$ $x_0 = 0,$ moving $U = U(x)$ with sufficient accuracy can be approximated by a linear function

$$U^{(k)}(x) = a_0^k + a_1^k \cdot x; \qquad x_{k-1} \le x \le x_k$$

Instead of the odds a_0^k, a_1^k introduce displacement values $U_k = U(x_k)$ ends of the plots, which will be the main unknowns. Obviously

$$U_{k-1} = a_0^k + a_1^k \cdot x_{k-1}; \quad U_k = a_0^k + a_1^k \cdot x_k$$

Where from

$$a_1^k = \frac{1}{x_k - x_{k-1}} \cdot (U_k - U_{k-1}); \qquad a_0^k = \frac{U_{k-1} \cdot x_k - U_k \cdot x_{k-1}}{x_k - x_{k-1}}$$

Thus, within $k^{-\omega}$ plot (element)

$$U^{(k)}(x) = \frac{1}{h_k}\left[U_{k-1}(x_k - x) + U_k(x_k - x_{k-1})\right] \tag{4}$$

$$h_k = x_k - x_{k-1}$$

Travel Continuity Conditions $U^{(k)}$ when passing from site to site, it leads to the following approximate representation of the function $U(x)$ on the whole segment $0 \le x \le l$

$$U(x) \approx U_h(x) \approx \sum_{k=1}^{n} U_k \varphi_k(x);$$

$$\varphi(x) = \begin{cases} \frac{1}{h_k}(x - x_{k-1}), & x_{k-1} \le x \le x_k \\ \frac{1}{h_k}(x_k - x), & x_k \le x \le x_{k+1} \\ 0, & x \notin [x_{k-1}, x_{k+1}] \end{cases} \tag{5}$$

Substituting (4) into (1), finding the surface forces at the ends of the sections, distributing the mass forces appropriately between the two ends (for example, consider that to each end of the section $[x_{k-1}, x_k]$ a force equal to half the resultant mass forces is applied) and making up the equilibrium conditions for all the forces applied to the cross-section $x_k = const$, let's get to the system n equations for n unknown

$$-\frac{ES_k}{h_k}(U_k - U_{k-1}) + \frac{ES_{k+1}}{h_{k+1}}(U_{k+1} - U_k) + \frac{1}{2}\int_{x_{k-1}}^{x_{k+1}} S(x)F(x)dx = 0 \tag{6}$$

$$U_0 = 0, \qquad S_{n+1} = 0$$

This is the idea of the finite element method in its original formulation; sections into which the rod is divided in combination with the law of distribution of the desired displacement field in these sections are called finite elements. Similarly, finite elements were constructed [1] for two-dimensional and three-dimensional problems; in contrast to the case considered, the one-dimensional region consists in the fact that instead of dividing the one-dimensional region into segments, we used the division of flat regions into triangles or quadrangles, spatial regions into tetrahedral or parallelepipeds, and instead of approximations (5), the functions of one variable were used piecewise linear approximations of functions of several variables [1].

It is not difficult to show that the system of the Ritz method using coordinate functions of the form (5) coincides with the system (6) in this problem. In fact, problem (2) and (3) is equivalent to the problem of minimum functional

$$J(U)=\frac{1}{2}\int_0^4 ES(x)\left(\frac{dU}{dx}\right)^2 dx-\int_0^4 S(x)F(x)U(x)dx \qquad (7)$$

Too many functions $H^1(O,L)$; satisfying the condition $U(0)=0$ and quadratic ally summable together with their first-order derivatives.

Considering functional (7) as a function in infinite-dimensional space $H^1(O,L)$, we write the necessary condition for the minimum of this function in the form of its differential being equal to zero

$$dJ=\frac{1}{2}\int_0^4 ES(x)\frac{dU}{dx}\cdot\frac{d\vartheta}{dx}dx-\int_0^4 S(x)F(x)\vartheta(x)dx=0 \qquad \vartheta\in H^1(O,L) \qquad (8)$$

Decision U equation (8) is sought in the form (5). Substituting in (8) alternately $\vartheta=\varphi_k$ we arrive at the following linear algebraic system with respect to U_k

$$\sum_{k=1}^{n}U_k\int_0^4 ES(x)\frac{d\varphi_k}{dx}\cdot\frac{d\varphi_1}{dx}dx-\int_0^4 S(x)F(x)\varphi_1(x)dx=0 \qquad (9)$$

I.e. $\varphi_1\neq 0$ only on the segment $[x_{k-1},x_k]$ and φ_k on the segment $[x_{k-1},x_k]$ it can be rewritten as

$$-\frac{U_{l-1}}{h_l}\int_{x_{l-1}}^{x_l}\frac{ES(x)}{h_l}dx+\frac{U_l}{h_l}\int_{x_{l-1}}^{x_l}\frac{ES(x)}{h_l}dx+\frac{U_l}{h_{l+1}}\int_{x_l}^{x_{l+1}}\frac{ES(x)}{h_{l+1}}dx,$$

$$-\frac{U_{l+1}}{h_{l+1}}\int_{x_l}^{x_{l+1}}\frac{ES(x)}{h_{l+1}}dx-\frac{U_l}{h_l}\int_{x_{l-1}}^{x_l}\frac{S(x)F(x)}{h_l}(x-x_{l-1})dx-\int_{x_l}^{x_{l+1}}\frac{S(x)F(x)}{h_{l+1}}(x_l-x)dx=0 \qquad (10)$$

Using the simplest quadrature formulas to calculate the integrals (10), we arrive at equation (6), which proves the statement that the finite element method in its simplest formulation coincides with the well-known Ritz method. In what follows, by the finite element method we mean the method of constructing coordinate functions of type (5) in the implementation of the Ritz method.

So, let there be an operator equation

$$AU=f \qquad (11)$$

Where A – positive definite self-adjoint operator acting on many functions $Д_A$ given in the two-dimensional region Ω with border Γ Euclidean space R^2 for example

$$A=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}, \qquad Д_A=\left\{U\big/U\in C^{(2)}(\Omega),U|_\Gamma=0\right\} \qquad (12)$$

As is known, the problem of solving equation (11) is equivalent to the problem of finding the minimum functional

$J(U)=(AU,U)-2(U,f)\equiv a(U,U)-2(U,f)$

The solution to the last problem exists and is unique if it is sought in space $H_0^1(\Omega)$ functions quadratic ally summable together with their first derivatives and vanish on the boundary.

Functional minimum conditions $J(U)$ we write in the form of equality to zero of the Freshet differential

$d[J(U)]=(AU,\vartheta)-(\vartheta,f)=0$

Thus, the problem of solving equation (11) is reduced to finding the element $U\in V$ such that a

$(U,\vartheta)=(f,\vartheta), \qquad \vartheta\in V$

For example, (12), equation (13) has the form

$$\int_\Omega gradU\cdot grad\vartheta d\Omega=\int_\Omega f\vartheta d\Omega \qquad \vartheta\in H_0^1(\Omega)$$

Instead of an exact solution U we will look for a solution U_h in some finite-dimensional subspace $V_h\in V$. An approximate problem is formulated as follows: find $U_h\in V_h$ such that

$$a(U_h,\vartheta_h)=(f,\vartheta_h) \qquad (13)$$

According to the finite element method, space V_h is constructed as follows. Let be h - numerical parameter, which we will tend to zero. We call triangulation τ_h areas of Ω its division into triangular subdomains.

$\overline{\Omega}=U_iK_i, \qquad K_i\in\tau_h$

We assume that the diameter of any K_i does not exceed h, two different triangles either do not have common points at all, or have one common vertex, or have one common side.

Space V_h for example (12) we define as the space of functions having the following properties:

A) narrowing of any function $\vartheta_h \in V_h$ to any triangle K_i is a polynomial not higher than the first;

B) ϑ_h непрерывна на $\overline{\Omega}$;

V) $\vartheta_h = 0$ на Γ.

Consider a triangle K_i, the radius vectors of the vertices of which are $\vec{a}_1, \vec{a}_2, \vec{a}_3$

If the values of some continuous in Ω the functions U at the tops are the essence U_1, U_2, U_3 then its interpolation by a first order polynomial in K_i has the form:

Where the functions are $\lambda_i(\vec{x})$ called the barycentric coordinates in the triangle K_i and are determined from the equations

$$\vec{x} = \lambda_1(\vec{x})\vec{a}_1 + \lambda_2(\vec{x})\vec{a}_2 + \lambda_3(\vec{x})\vec{a}_3$$
$$1 = \lambda_1(\vec{x}) + \lambda_2(\vec{x}) + \lambda_3(\vec{x})$$

Combining the interpolations (14) of the function over all triangles, we obtain a piecewise linear $U(\vec{x})$ in the whole area Ω. So any function $U_h = V_h$ is determined uniquely by its value at the nodes-vertices of triangles and, therefore, V_h has finite dimension. $N(h)-$ Total number of nodal unknowns. The basis in this space is constructed as follows. Let be $\{\vec{a}_i\}_{i=1}^{N(h)}$ the set of all internal nodes (i.e., nodes not lying on the boundary), and let $W_{i,h}$ a function that is a polynomial not higher than the first in each of K_i besides

$$W_{i,h}(\vec{a}_j) = \delta_{i,j}, \qquad \delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Thus, in each triangle three basic functions will be nonzero, the values of which at each point $\vec{x}$ the triangle K_i will coincide with the barycentric coordinates of the point $\vec{x}$ in this triangle.

Every function $\vartheta_h \in V_h$ can be represented as

$$\vartheta_h = \sum_{i=1}^{N(h)} \vartheta_h(\vec{a}_i) \cdot W_{i,h}(\vec{x})$$

And this view is unique.

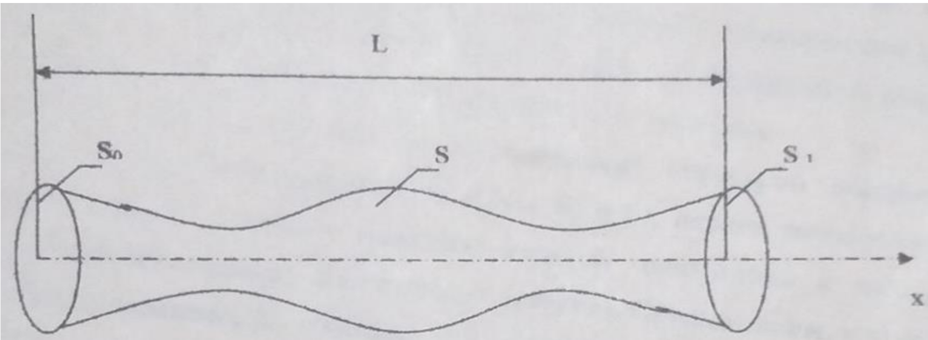

Fig.1

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