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Shreemathi Adiga
Assistant Professor,
Department of Mathematics,
Government First Grade
College, Koteshwara,
Kundapura Taluk, Udupi,
Karnataka, India

N. Anusheela
Assistant Professor,
Department of Mathematics,
Government Arts
College,Udhagamandalam, the
Nilgiris, India

M.A. Gopalan
Professor, Department of
Mathematics, Shrimati Indira
Gandhi College, Trichy,
Tamil Nadu, India

Correspondence:
Shreemathi Adiga
Assistant Professor,
Department of Mathematics,
Government First Grade
College, Koteshwara,
Kundapura Taluk, Udupi,
Karnataka, India

Observations on Pell-like Equation $11x^2-6y^2=-10$

Shreemathi Adiga, N. Anusheela, M.A. Gopalan

Abstract
This paper concerns with the problem of finding non-zero distinct integer solutions to the Pell-like equation represented by $11x^2-6y^2=-10$. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions

Introduction
The diophantine equation offersa field for research due to their variety [1-3].In particular, the binary quadratic diophantine equations of the form $ax^2-by^2=N$, $(a,b,N\neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [4-16].
This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation representing hyperbola given by $11x^2-6y^2=-10$. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed.

Notations
- Polygonal number of rank n with size m

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

- Pyramidal number of rank n with size m

$$P_n^m=\frac{1}{6}[n(n+1)][(m-2)n+(5-m)]$$

Method of Analysis:
The Diophantine equation under consideration is

$$11x^2-6y^2=-10 \tag{1}$$

Taking

$$x=X+6T\,,\ y=X+11T \tag{2}$$

In (1), it reduces to the equation

$$X^2=66T^2-2 \tag{3}$$

The smallest positive integer solution (T_0,X_0) of (3) is
$T_0=1\,,\ X_0=8$

To obtain the other solutions of (3), consider the Pellian equation

$$X^2 = 66T^2 + 1 \quad (4)$$

Whose smallest positive integer solution is

$$\tilde{T}_0 = 8\,,\ \tilde{X}_0 = 65$$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (4) is given by

$$\tilde{X}_n + \sqrt{66}\tilde{T}_n = (65 + 8\sqrt{66})^{n+1}, n = 0,1,2,\ldots \quad (5)$$

Since irrational roots occur in pairs, we have

$$\tilde{X}_n - \sqrt{66}\tilde{T}_n = (65 - 8\sqrt{66})^{n+1}, n = 0,1,2,\ldots \quad (6)$$

From (5) and (6), solving for $\tilde{X}_n, \tilde{T}_n$, we have

$$\tilde{X}_n = \frac{1}{2}\left[(65 + 8\sqrt{66})^{n+1} + (65 - 8\sqrt{66})^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{66}}\left[(65 + 8\sqrt{66})^{n+1} - (65 - 8\sqrt{66})^{n+1}\right] = \frac{1}{2\sqrt{66}} g_n$$

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the general solution (T_{n+1}, X_{n+1}) of (3) is found to be

$$T_{n+1} = \frac{4}{\sqrt{66}} g_n + \frac{1}{2} f_n$$

$$X_{n+1} = 4 f_n + \frac{33}{\sqrt{66}} g_n$$

In view of (2), we have

$$x_{n+1} = 7 f_n + \frac{57}{\sqrt{66}} g_n$$

$$y_{n+1} = \frac{19}{2} f_n + \frac{37}{\sqrt{66}} g_n$$

A few numerical examples are given in the following Table: 1

Table 1: Numerical Example

n	x_{n+1}	y_{n+1}
-1	14	19
0	1822	2467
1	236846	320691
2	30788158	41687363
3	4002223694	5419036499

Note that the $x-$ values even and $y-$ values are odd.

Recurrence relations for x and y are:

$$x_{n+3} - 130x_{n+2} + x_{n+1} = 0, n = -1,0,1.....$$

$$y_{n+3} - 130y_{n+2} + y_{n+1} = 0, n = -1,0,1.....$$

A few interesting relations among the solutions are given below:

- $240y_{n+1} = 5x_{n+2} - 325x_{n+1}$
- $48y_{n+2} = 65x_{n+2} - x_{n+1}$
- $1672x_{n+1} = 19y_{n+2} - 1235y_{n+1}$
- $1672x_{n+2} = 1235y_{n+2} - 19y_{n+1}$

➢ Each of the following expressions represents a cubical integer.

- $\frac{3}{40}\left[19x_{n+2} - 2467x_{n+1}\right] + \frac{1}{40}\left[19x_{3n+4} - 2467x_{3n+3}\right]$
- $\frac{3}{20}\left[911y_{n+1} - 7y_{n+2}\right] + \frac{1}{20}\left[911y_{3n+3} - 7y_{3n+4}\right]$
- $\frac{3}{325}\left[114y_{n+2} - 20042x_{n+1}\right] + \frac{1}{325}\left[114y_{3n+4} - 20042x_{3n+3}\right]$
- $\frac{6}{325}\left[7401y_{n+1} - 77x_{n+2}\right] + \frac{2}{325}\left[7401y_{3n+3} - 77x_{3n+4}\right]$

➢ Each of the following expressions represents bi-quadratic integer:

- $\frac{1}{40}\left[240 + 76x_{2n+3} - 9868x_{2n+2} + 19x_{4n+5} - 2467x_{4n+4}\right]$
- $\frac{1}{20}\left[120 + 3644y_{2n+2} - 28y_{2n+3} + 911y_{4n+4} - 7y_{4n+5}\right]$
- $\frac{1}{325}\left[1950 + 456y_{2n+3} - 80168x_{2n+2} + 114y_{4n+5} - 20042x_{4n+4}\right]$
- $\frac{1}{325}\left[1950 + 59208y_{2n+2} - 616x_{2n+3} + 14802y_{4n+4} - 154x_{4n+5}\right]$

➢ Each of the following expressions represents Nasty number:

➢

- $\frac{1}{20}\left[240 + 57x_{2n+3} - 7401x_{2n+2}\right]$
- $\frac{1}{10}\left[120 + 2733y_{2n+2} - 21y_{2n+3}\right]$
- $\frac{1}{325}\left[3900 + 684y_{2n+3} - 120252x_{2n+2}\right]$
- $\frac{1}{325}\left[3900 + 88812y_{2n+2} - 924x_{2n+3}\right]$

Remarkable Observations

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table: 2 below

Table 2: Hyperbolas

S.No	Hyperbolas	(X,Y)
1	$9X^2-66Y^2=57600$	$(19x_{n+2}-246x_{n+1}, 911x_{n+1}-7x_{n+2})$
2	$484X^2-23826Y^2=69889600$	$(17309y_{n+1}-133y_{n+2}, 19y_{n+2}-2467y_{n+1})$
3	$4X^2-66Y^2=422500$	$(7401y_{n+1}-77x_{n+2}, 19x_{n+2}-1822y_{n+1})$
4	$X^2-66Y^2=422500$	$(114y_{n+2}-20042x_{n+1}, 2467x_{n+1}-14y_{n+2})$

2. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table: 3 below:

Table 3: Parabolas

S.No	Parabolas	(X,Y)
1	$180X-33Y^2=14400$	$(19x_{2n+3}-2467x_{2n+2}\ ,\ 911x_{n+1}-7x_{n+2})$
2	$4840X-33Y^2=193600$	$(911y_{n+2}-7y_{2n+3}\ ,\ 19y_{n+2}-2467y_{n+1})$
3	$650X-66Y^2=211250$	$(7401y_{2n+2}-77x_{2n+3}\ ,\ 19x_{n+2}-1822y_{n+1})$
4	$325X-66Y^2=211250$	$(114y_{2n+3}-20042x_{2n+2}\ ,\ 24767x_{n+1}-14y_{n+2})$

3. Let p,q be two non-zero distinct integers such that $p>q>0$. Treat p,q as the generators of the Pythagorean triangle $T(\alpha,\beta,\gamma)$ where $\alpha=2pq, \beta=p^2-q^2, \gamma=p^2+q^2, p>q>0$

Taking $p=x_{n+1}+y_{n+1}, q=x_{n+1}$, it is observed that $T(\alpha,\beta,\gamma)$ is satisfied by the following relations:

- ❖ $11\alpha-3\beta-8\gamma=10$
- ❖ $14\alpha-11\gamma-10=\frac{12A}{P}$
- ❖ Each of the following is a Nasty number:
  - ✓ $3\left(\alpha-\frac{4A}{P}\right)$
  - ✓ $11(\gamma-\alpha)+10$

Where A,P represent the area and perimeter of $T(\alpha,\beta,\gamma)$.

4. Relations between solutions and special polygonal numbers:

- ❖ $11\left(P_x^5*t_{3,y+1}\right)^2-54\left(P_y^3*t_{3,x}\right)^2=-10\left(t_{3,x}*t_{3,y+1}\right)^2$
- ❖ $99\left(P_x^3*t_{3,y}\right)^2-6\left(P_y^5*t_{3,x+1}\right)^2=-10\left(t_{3,x+1}*t_{3,y}\right)^2$
- ❖ $11\left(P_x^5*t_{3,2y-2}\right)^2-216\left(P_{y-1}^4*t_{3,x}\right)^2=-10\left(t_{3,x}*t_{3,2y-2}\right)^2$
- ❖ $99\left(P_x^3*t_{3,y-1}\right)^2-6\left(t_{3,x+1}\left(P_{y-1}^5+t_{3,y-1}\right)\right)^2=-10\left(t_{3,x+1}*t_{3,y-1}\right)^2$
- ❖ $99\left(P_x^3*P_{y+1}^5\right)^2-6\left(t_{3,x+1}*t_{3,y^2+2y}\right)^2=-10\left(t_{3,x+1}*P_{y+1}^5\right)^2$

Conclusion

In this paper, we have presented infinitely many integer solutions for the diophantine equation representing hyperbola is given by $11x^2-6y^2=-10$. As the binary quadratic diophantine equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties.

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