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Some Properties of Strongly Nil Clean Elements

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Abstract
The ring concept has many developments, especially whose element is the sum of some specific elements in that ring, such as clean ring and nil clean ring, they are developed into strongly clean ring and strongly nil clean ring by adding commutative property to multiplication operation on those specific elements in the rings. This research is developed from those concepts by investigating some properties of strongly nil clean elements and its connection with strongly clean elements, strongly nil clean ideals, and strongly nil cleanness of elements in direct product rings.

Keywords: Strongly clean, strongly nil clean, idempotent, nilpotent.

Introduction

In this article, R is a unital ring. The set of all units, the set of all idempotents, and the set of all nilpotents of a ring R are denoted by $U(R), Id(R)$, and $Nil(R)$, respectively. According to [1], an element r in R is called clean element if it can be expressed by $r = u + e$, for some u in $U(R)$ and e in $Id(R)$. A ring R is called clean ring if every element in R is clean. The concept of clean element has motivated other researchers to develop other decomposition elements such as strongly clean element ([2], [3], and [4]), weakly clean element ([5] and [6]), which the decomposition of this element using idempotent as one of its summand elements. Moreover, replacing idempotent elements by nilpotent elements, some research are developed by investigating nil clean elements ([8] and [9]), and also some specific subrings such as clean ideals [10] and nil clean ideals ([11]). Motivated by these ideas, in this articles, it will be discussed some properties of strongly nil clean elements and its connection with strongly clean elements, strongly nil clean ideals, and strongly nil cleanness of elements in finite direct product rings.
The following lemma is provided to be used in the next theorems with the proof refer to [7] and [9].

Lemma 1

a. If e is an idempotent element in R, then so is $1 - e$
b. If n is a nilpotent element in R, then $n + 1 \in U(R)$
c. If R commutative and $a, b \in Nil\,(R)$, then $a + b, ab \in Nil\,(R)$
d. If $e \in Id(R)$ and $n \in Nil\,(R)$ then $2e - 1 + n \in U(R)$.

Let R be a unital ring. A subring I of R is called strongly clean ideal if I is an ideal and any $a \in I$ can be expressed as $a = e + u$, for some $e \in Id(R)$, $u \in U(R)$ and $eu = ue$. A subring I of R is called strongly nil clean ideal if I is an ideal and any $a \in I$ can be expressed as $a = e + n$, for some $e \in Id(R)$, $n \in Nil(R)$ and $en = ne$, in the case the decomposition is unique, I is called uniquely strongly nil clean ideal. The following theorem showed the connection between strongly nil clean ideal and strongly clean ideal.

Theorem 2

Let R be a unital ring and I is an ideal of R. If I is strongly nil clean, then I is strongly clean.

Proof

For any $a \in I$, we get $-a \in I$. By definition of strongly nil clean ideal, there exist $e \in Id(R)$ and $n \in Nil\,(R)$ such that $-a = e + n$ and $en = ne$.

Since $1 - e \in Id(R)$ by lemma 1(a) and $-(1+n) \in U(R)$ by lemma 1(b), the element a can be written as

$a = -(-a) = -(e+n) = (1-e) + (-1-n) \ldots\ldots\ldots (*)$

Now, by commuteness of $en = ne$, consider that

$$\begin{aligned}(1-e)(-1-n) &= -1 - n + e + en \\ &= -1 - n + e + ne \\ &= (-1-n)(1-e) \ldots\ldots\ldots (**)\end{aligned}$$

From (*) and (**), it is concluded that a is strongly clean element. It means any strongly nil clean ideal is strongly clean ideal.

Note that the converse of Theorem 2 is not satisfied as the following counter example.

Example 3

In ring $\mathbb{Z}_{10}$, we see $I = 2\mathbb{Z}_{10} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\}$ is a strongly clean ideal but not strongly nil clean. Since $Id(\mathbb{Z}_{10}) = \{\bar{0}, \bar{1}, \bar{5}, \bar{6}\}$, $U(\mathbb{Z}_{10}) = \{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$, and $Nil(\mathbb{Z}_{10}) = \{\bar{0}\}$, it is obtained that $U(\mathbb{Z}_{10}) + Id(\mathbb{Z}_{10}) = I$, but $Nil(\mathbb{Z}_{10}) + Id(\mathbb{Z}_{10}) = Id(\mathbb{Z}_{10}) \neq I$.

The following theorem showed some properties of strongly nil cleanness related to its homomorphic images.

Theorem 4

Let R be a unital ring with a nil ideal I of R. If $\emptyset$ is canonical homomorphism from R to R/I by $\emptyset$(r)=$\bar{r}$=r+I for any r in R, then

1. If a is strongly nil clean element in R, then $\emptyset$(a) also strongly nil clean in R/I.
2. If I is strongly nil clean ideal, then $\emptyset(I)$ also strongly nil clean ideal.
3. In the case R is commutative, for any strongly nil clean element $\bar{y}$ in R/I, there exists a strongly nil clean element x in R such that $\bar{x} = \bar{y}$.

Proof

1. By its definition, any strongly nil clean element a can be expressed by $a = e + n$, for some $e \in Id(R)$ and $n \in Nil\,(R)$ with $en = ne$.
   Since $\emptyset$ is homomorphic, it yields $\emptyset(a) = \emptyset(e+n) = \emptyset(e) + \emptyset(n)$.
   It is obviously that $\emptyset(e) \in Id(R/I)$, since e is idempotent and
   $[\emptyset(e)]^2 = (e+I)^2 = e^2 + I = e + I = \bar{e} = \emptyset(e)$.

   Meanwhile, $n \in Nil(R)$ means there exists some positive integer k such that $n^k = 0$.

   Consider that $[\emptyset(n)]^k = \emptyset(n^k) = \emptyset(0) = \bar{0}$, which shows that $\emptyset(n)$ is nilpotent in R/I.
   Finally, the condition $en = ne$ implies
   $$\emptyset(e)\emptyset(n) = \emptyset(en) = \emptyset(ne) = \emptyset(n)\emptyset(e)$$
   So, it concludes that $\emptyset(a)$ is strongly nil clean.
2. For any $\bar{z}$ in $\emptyset(I)$, there exists a in I such that $\bar{z} = \emptyset(a) = a + I$.
   Since I is strongly nil clean ideal, a can be decomposed as $a = e + n$, for some $e \in Id(R)$ and $n \in Nil\,(R)$ with $en = ne$. Therefore, $\bar{z} = a + I = (e+n) + I = (e+I) + (n+I) = \emptyset(e) + \emptyset(n)$
   By the proof (1), it is showed that $\emptyset(e) \in Id(R/I)$ and $\emptyset(n) \in Nil(R/I)$, with $\emptyset(e)\emptyset(n) = \emptyset(n)\emptyset(e)$.
   So, $\emptyset(I)$ is strongly nil clean ideal of R/I.
3. For any strongly nil clean element $\bar{y}$ in R/I, there exists $\bar{f} \in Id(R/I)$ and $\bar{m} \in Nil(R/I)$, such that
   $$\bar{y} = \bar{f} + \bar{m} \text{ with } \bar{f}\bar{m} = \bar{m}\bar{f}.$$
   By proposition 1 in [12], for $\bar{f} \in Id(R/I)$ there exists an idempotent e in R such that $\bar{e} = \bar{f}$.
   Next, $\bar{m} = m + I$ is nilpotent. It means there exists some positive integer k such that $(\bar{m})^k = \bar{0} = I$ in R/I.
   So, $m^k + I = (m+I)^k = (\bar{m})^k = \bar{0} = I$, which follows $m^k \in I$ and implies $(m^k)^l = 0$ for some positive integer l (since I is nil ideal). It concludes m is nilpotent.
   From $e \in Id(R)$ and $m \in Nil\,(R)$, we get a srongly nil clean element $x = e + m$ in R such that
   $$\bar{x} = \overline{e+m} = \bar{e} + \bar{m} = \bar{f} + \bar{m} = \bar{y}\,.$$

From two rings R and S, it can be constructed a new ring (it is called direct product of R and S) under additive and multiplication operations on $T = R \times S = \{(a,b) : a \in R, b \in S\}$, that are

$(a,b) + (c,d) = (a + c.b + d)$ and $(a,b) \cdot (c,d) = (ac, bd)$, for any $(a,b), (c,d) \in T$.

Theorem 5

Let $R_1, R_2, R_3, \ldots, R_n$ be unital rings and $R = R_1 \times R_2 \times R_3 \times \ldots \times R_n$. It follows that $a = (a_1, a_2, \ldots, a_n)$ is strongly nil clean element in R if and only if a_i is strongly nil clean element in R_i for all $i = 1,2,..,n$.

Proof

($\Rightarrow$) Let $a = (a_1, a_2, \ldots, a_n)$ be any strongly nil clean element in R.

It means, there is decomposition $a = e + f$, for some $e \in Id(R)$, and $f \in Nil(R)$ with $ef = fe$.

Let $e = (e_1, e_2, \ldots, e_n)$ and $f = (f_1, f_2, \ldots, f_n)$.

From $a = e + f$, it provides

$$\begin{aligned}(a_1, a_2, \ldots, a_n) &= (e_1, e_2, \ldots, e_n) + (f_1, f_2, \ldots, f_n) \\ &= (e_1 + f_1, e_2 + f_2, \ldots, e_n + f_n),\end{aligned}$$

or equivalently $a_i = e_i + f_i$, for all $i = 1,2,..n$

Since e is idempotent, it is easy to observe that

$$\begin{gathered} e^2 = e \\ \Leftrightarrow (e_1, e_2, \ldots, e_n)^2 = (e_1, e_2, \ldots, e_n) \\ \Leftrightarrow ({e_1}^2, {e_2}^2, \ldots, {e_n}^2) = (e_1, e_2, \ldots, e_n) \\ \Leftrightarrow {e_i}^2 = e_i, \text{ for all } i = 1,2,\ldots,n. \\ \Leftrightarrow e_i \in Id(R_i), \text{for all } i = 1,2,\ldots,n. \end{gathered}$$

Meanwhile, $f = (f_1, f_2, \ldots, f_n)$ nilpotent in R implies there exists positive integers m such that

$$\begin{gathered} f^m = 0 \\ \Leftrightarrow (f_1, f_2, \ldots, f_n)^m = (0_1, 0_2, \ldots, 0_n) \\ \Leftrightarrow ({f_1}^m, {f_2}^m, \ldots, {f_n}^m) = (0_1, 0_2, \ldots, 0_n) \\ \Leftrightarrow {f_i}^m = 0_i, \textit{for all}\, i = 1,2,\ldots,n; \end{gathered}$$

which means $f_i \in Nil(R_i)$, for all $i = 1,2,\ldots,n$.

Finally, the condition $ef = fe$ provides

$$\begin{gathered} (e_1, e_2, \ldots, e_n)(f_1, f_2, \ldots, f_n) = (f_1, f_2, \ldots, f_n)(e_1, e_2, \ldots, e_n) \\ (e_1 f_1, e_2 f_2, \ldots, e_n f_n) = (f_1 e_1, f_2 e_2, \ldots, f_n e_n) \\ e_i f_i = f_i e_i, \textit{for all } i = 1,2,\ldots,n. \end{gathered}$$

So, in R_i, it follows that $a_i = e_i + f_i$, for all $i = 1,2,\ldots,n$ are strongly nil clean elements.

(⇐) Conversely, let a_i be strongly nil clean elements in R_i, for all $i = 1,2, \dots, n$.

It means, $a_i = e_i + f_{i,}$ for some $e_i \in Id(R_i)$ and $f_i \in Nil(R_i)$ and $e_i f_{i,} = f_i e_i$.

It is obviously that $e = (e_1, e_2, \dots, e_n) \in Id(R)$.

Now, $f_i \in Nil(R_i)$ means that there exist positive integers $k_1, k_2, \dots, k_n$ such that

$$f_i^{k_i} = 0_i, \text{for all } i = 1, \dots, n.$$

By choosing, $k = \max\{k_1, k_2, \dots, k_n\}$, it is obtained

$f_i^{\ k} = 0_i$, for all $i = 1,2, \dots, n$

Therefore if $f = (f_1, f_2, \dots, f_n)$ in R, then|

$$\begin{aligned} f^k &= (f_1, f_2, \dots, f_n)^k \\ &= (f_1^{\ k}, f_2^{\ k}, \dots, f_n^{\ k}) \\ &= (0_1, 0_2, \dots, 0_n), \end{aligned}$$

which means $f \in Nil(R)$.

Finally, the conditions $e_i f_i = f_i e_i$ for all $i = 1,2, \dots, n$ implies that $ef = fe$.

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