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Special Pythagorean Triangles with Sum of Two Legs a Hexic

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Abstract
Pythagorean triangles have been discovered satisfying different constraints. In this paper Pythagorean Triangles (X, Y, Z) are found with perimeter as sum of two squares and a hexic using the software Mathematica.

Keywords: Euclidean formula, Opposite Parity, Primitive Pythagorean Triangle, Diophantine Equation.

Introduction

Gopalan and Vijayasankar [1] have determined the solutions of hexic Diophantine equation. Some very Special Pythagorean Triangles with their perimeters as both Triangular and Pentagonal numbers have been found by Darbari [2]. Some Perfect Pythagorean Triangles where their perimeters are Quaternary numbers are also discovered by Darbari [3]. In this paper, an effort has been made to find Special Pythagorean Triangles with perimeter as sum of two squares and a hexic. *Such triangles with two* sides consecutive are also investigated. Various 3D graphs are also plotted.

Method of Analysis

By Euclidean formula, the primitive solutions of the Pythagorean Equation,

$X^2 + Y^2 = Z^2$, **(1)**

are given by [4], where

$X = m^2 - n^2,\ Y = 2mn,\ Z = m^2 + n^2$, **(2)**

for some integers *m*, *n* of opposite parity such that $m > n > 0$ and $(m, n) = 1$.

***Sum of two legs is a hexic:** In that case,

$X + Y = \beta^6, \quad \beta \in N$, **(3)**

i.e., $m^2 + 2mn - n^2 - \beta^6 = 0$. **(4)**

The solutions of equation (4) are given, by theory of equations, as

$\beta = -(m^2 + 2mn - n^2)^{1/6}$,
$\beta = (m^2 + 2mn - n^2)^{1/6}$,
$\beta = -(-1)^{1/3}(m^2 + 2mn - n^2)^{1/6}$,
$\beta = (-1)^{1/3}(m^2 + 2mn - n^2)^{1/6}$,
$\beta = -(-1)^{2/3}(m^2 + 2mn - n^2)^{1/6}$,
$\beta = (-1)^{2/3}(m^2 + 2mn - n^2)^{1/6}$. **(5)**

Only the positive integral values of β are taken as the solutions.

***Perimeter is sum of two squares and a hexic:** If $X + Y = \beta^6$ then from equation (2), we have

$X + Y + Z = \beta^6 + m^2 + n^2$. **(6)**

Equations (2) and (4) generate (X, Y, Z) which satisfy equations (1) and (3) in correspondence with (6).

Solving the equation (4), using software *Mathematica*,

$m^2 + 2mn - n^2 - \beta^6 = 0$, for $n < m$; $0 < n < 10^9$; $0 < m < 10^9$; $0 < \beta < 10^9$; we get the 2685 values for (X, Y, Z). Out of these 2685 values for (X, Y, Z), only 136 are primitives. Following are the few examples in Table 1 (precisely 65) with $0 < \beta < 10^2$:-

Table 1: Special Pythagorean Triangles (X, Y, Z).

S.N.	M	n	β	$X = m^2 - n^2$	Y = 2mn	$Z = m^2 + n^2$
1	245	196	7	21609	96040	98441
2	265	114	7	57229	60420	83221
3	287	70	7	77469	40180	87269
4	3485	3094	17	2572389	21565180	21718061
5	3757	1734	17	11108293	13029276	17121805
6	4033	1136	17	14974593	9162976	17555585
7	6615	5292	21	15752961	70013160	71763489
8	7155	3078	21	41719941	44046180	60668109
9	7749	1890	21	56474901	29291220	63619101
10	8947	5474	23	50084133	97951756	110013485
11	8993	5290	23	52889949	95145940	108858149
12	12013	156	23	144287833	3748056	144336505
13	21359	16368	31	188295457	699208224	724118305
14	21449	15738	31	212374957	675128724	707744245
15	24025	7688	31	518095281	369408400	636305969
16	30625	24500	35	337640625	1500625000	1538140625
17	33125	14250	35	894203125	944062500	1300328125
18	35875	8750	35	1210453125	627812500	1363578125
19	48749	47068	41	161068377	4589035864	4591861625
20	48865	43818	41	467771101	4282333140	4307805349
21	66625	2378	41	4433235741	316868500	4444545509
22	73555	67116	47	905780569	9873434760	9914895481
23	77897	41064	47	4381690513	6397524816	7754194705
24	81733	30926	47	5723865813	5055349516	7636700765
25	83221	80030	49	520933941	13320353260	13330535741
26	84035	67228	49	2542277241	11299009960	11581485209
27	87269	49980	49	5117877961	8723409240	10113878761
28	90895	39102	49	6732934621	7108352580	9790867429
29	98441	24010	49	9114150381	4727136820	10267110581
30	105175	14168	49	10861048401	2980238800	11262512849
31	94095	83538	51	1875271581	15721016220	15832466469
32	101439	46818	51	8097945597	9498342204	12481795845
33	108891	30672	51	10916478297	6679809504	12798021465
34	178605	142884	63	11483908569	51039593640	52315583481
35	193185	83106	63	30413836989	32109665220	44227051461
36	209223	51030	63	41170202829	21353299380	46378324629
37	241569	147798	69	36511332957	71406830124	80199830565
38	242811	142830	69	38556772821	69361390260	79357590621
39	324351	4212	69	105185830257	2732332824	105221312145
40	256789	195300	71	27798500521	100301783400	104082680521
41	274415	124392	71	59830222561	68270061360	90776961889
42	307501	60492	71	90897582937	37202700984	98216147065
43	282437	191844	73	42966538633	108367687656	116574779305
44	299665	131546	73	72494762109	78839464180	107103462341
45	343705	52274	73	115400555949	35933670340	120865698101
46	326095	260876	77	38281661649	170140718440	174394236401
47	352715	151734	77	101384664469	107037715620	147431077981
48	381997	93170	77	137241059109	71181320980	154602356909
49	359371	236052	79	73426968937	169660486584	184868062345
50	405665	112338	79	151944265981	91143189540	177183918469
51	435625	386750	85	40193578125	336955937500	339344703125
52	469625	216750	85	173567078125	203582437500	267528203125
53	504125	142000	85	233978015625	143171500000	274306015625
54	514865	332682	89	154408655101	342572635860	375763281349
55	533021	266110	89	213296854341	283684436620	354925918541
56	652405	57188	89	422361816681	74619474280	428902751369
57	538265	430612	91	104302515681	463566736360	475155904769
58	582205	250458	91	276233452261	291635799780	401691871789
59	630539	153790	91	373928066421	193941185620	421230794621
60	576693	441936	93	137267388153	509722795296	527882244345
61	579123	424926	93	154821343653	492168839796	515945554605
62	648675	207576	93	377691459849	269298723600	463867051401
63	669593	417130	97	274357348749	558614656180	622352222549
64	719837	268884	97	445866701113	387105303816	590463912025

65	799765	131726	97	622272316149	210699688780	656975794301

The following Table 2 verifies that these are Pythagorean Triangles for first seven (X, Y, Z) in Table 1while Table 3 verifies that these Pythagorean Triangles (X, Y, Z) in Table 2 have perimeters as sum of two squares and a hexic.

Table 2: Verification of $X^2 + Y^2 = Z^{2.}$

S.N.	X^2	Y^2	$X^2 + Y^2$	Z^2
1	466948881	9223681600	9690630481	9690630481
2	3275158441	3650576400	6925734841	6925734841
3	6001445961	1614432400	7615878361	7615878361
4	6617185167321	465056988432400	471674173599721	471674173599721
5	123394173373849	169762033084176	293156206458025	293156206458025
6	224238435515649	83960129176576	308198564692225	308198564692225
7	248155780267521	4901842573185600	5149998353453121	5149998353453121

Table 3: $X + Y = \beta^6$ and $X + Y + Z = \beta^6 + m^2 + n^{2..}$

S.N.	$X + Y = \beta^6$	$X + Y + Z = \beta^6 + m^2 + n^2$
1	$117649 = 7^6$	$216090 = 7^6 + 245^2 + 196^2$
2	$117649 = 7^6$	$200870 = 7^6 + 265^2 + 114^2$
3	$117649 = 7^6$	$204918 = 7^6 + 287^2 + 70^2$
4	$24137569 = 17^6$	$45855630 = 17^6 + 3485^2 + 3094^2$
5	$24137569 = 17^6$	$41259374 = 17^6 + 3757^2 + 1734^2$
6	$24137569 = 17^6$	$41693154 = 17^6 + 4033^2 + 1136^2$
7	$85766121 = 21^6$	$157529610 = 21^6 + 6615^2 + 5292^2$

***One leg and hypotenuse are consecutive:**

When one leg and hypotenuse in a Pythagorean triangle are consecutive, then,

$m = n + 1.$ **(7)**

Substituting the value of m from equation (7) in equations (2) and (4), we get the Diophantine equation,

$2n^2 + 4n + 1 - \beta^6 = 0.$ **(8)**

Using the software Mathematica, equation (8) was solved for $0 < n < 10^{13}$ and $0 < \beta < 10^5$ and it was found that it has no solution. It needs further investigations to find whether any solution of (8) exists for $n \geq 10^{13}$.

3D Plots

For $n < m$, $0 < n < 10^9$, $0 < m < 10^9$, $0 < \beta < 10^9$, we get the 2685 values for (X, Y, Z) by solving equation (4) as obtained above. Plotting these values as ListPlot3D, ListSurfacePlot3D and ListPointPlot3D, we get figure 1, Figure 2 and Figure 3 respectively. They are as follows:

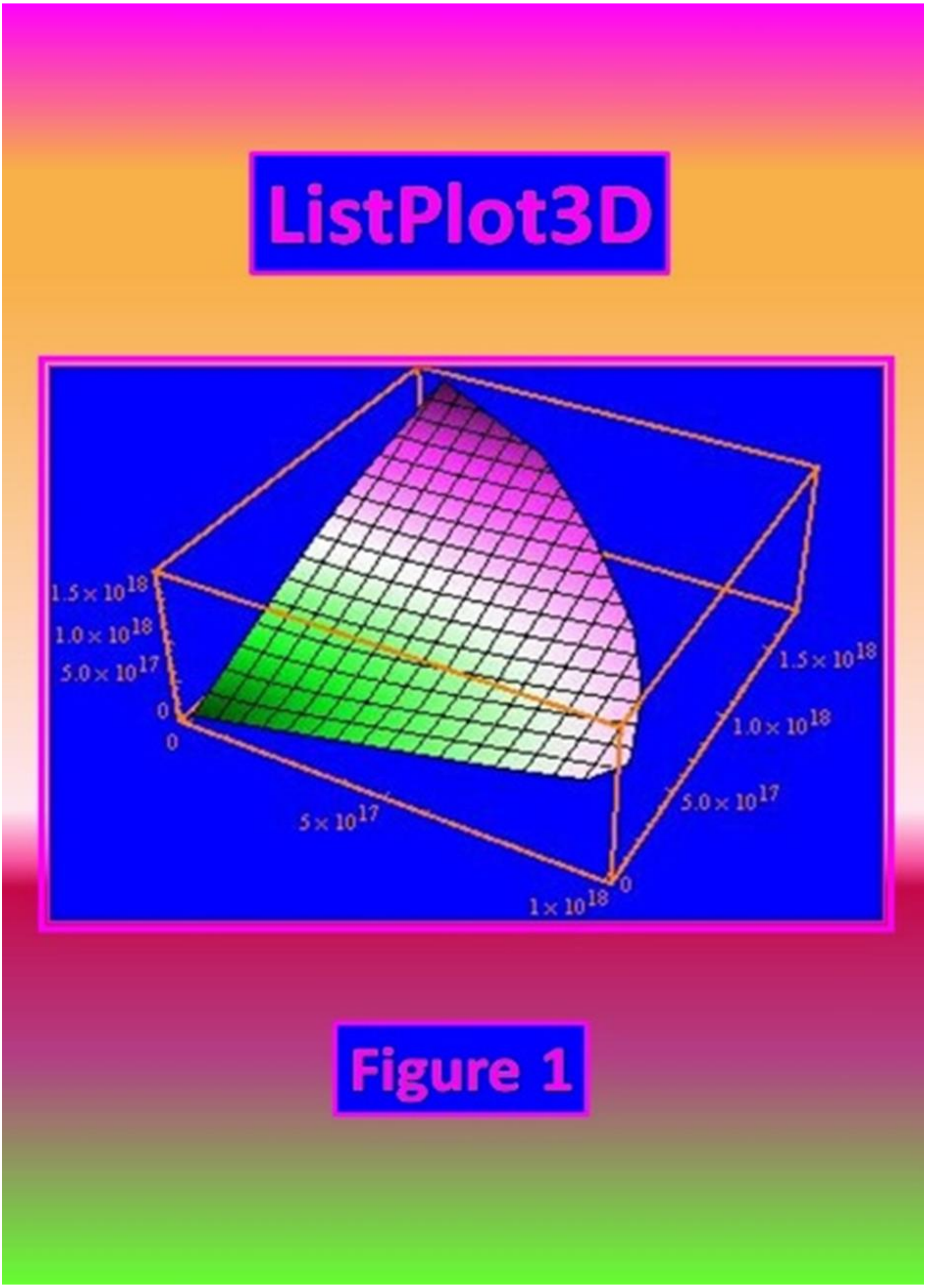

Figure 1

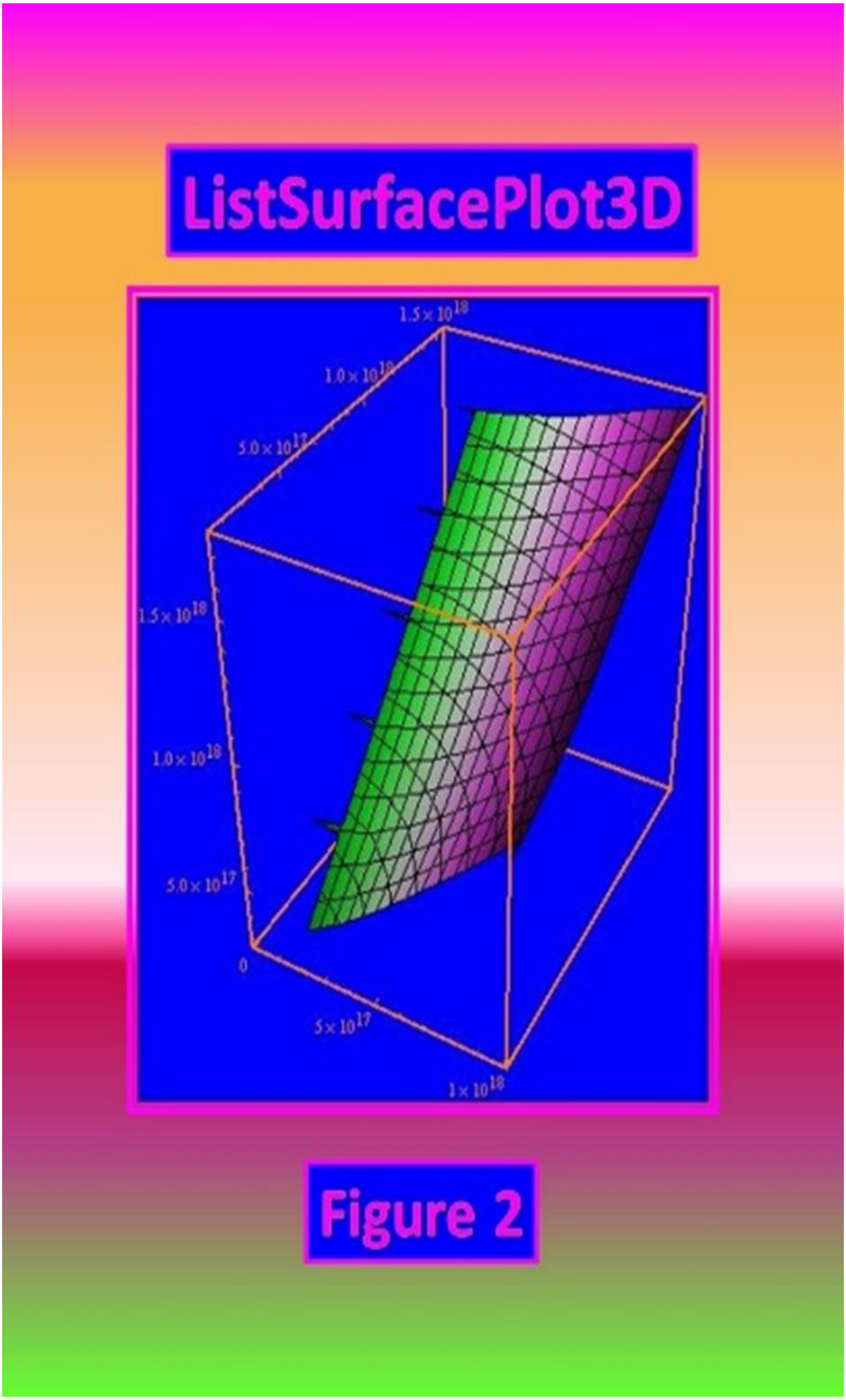
ListSurfacePlot3D
1.5 × 10^18
1.0 × 10^18
5.0 × 10^17
1.5 × 10^18
1.0 × 10^18
5.0 × 10^17
0
5 × 10^17
1 × 10^18

Figure 2

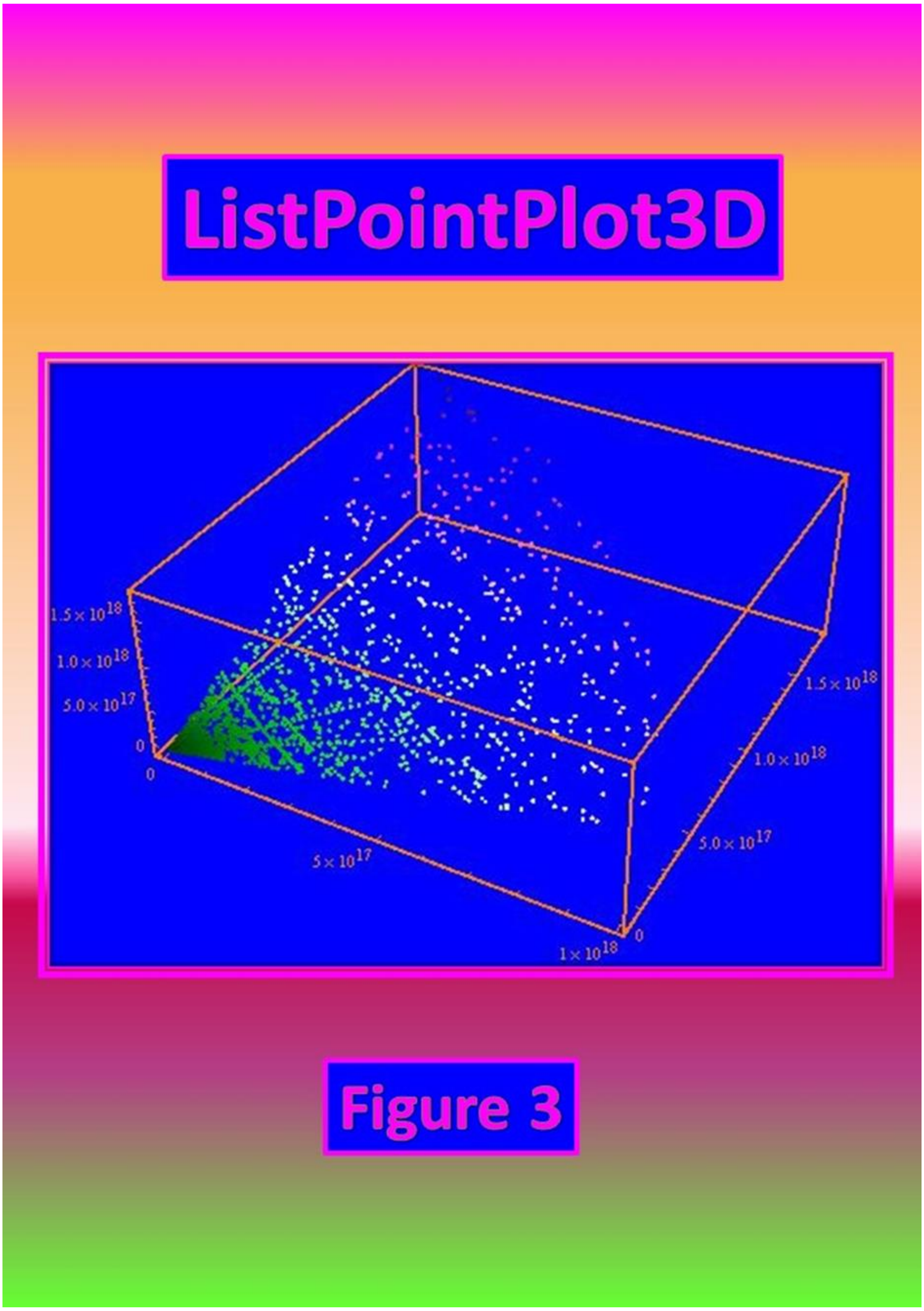

Figure 3

Observations

The following observations are made for $\beta < 100$: -

1. Except for $\beta = 49$ and $\beta = 98$, there are three Pythagorean Triangles corresponding to each β.
2. For $\beta = 49$ and $\beta = 98$, there are six Pythagorean Triangles satisfying $X + Y = \beta^6$.
3. $Y + Z - X = 0 (\text{mod } 2)$.

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